Robust Linear Quadratic Regulator의 설계

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Synthesis of Robust Linear Quadratic Regulator

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본 연구는 LQR 을 Robust 하게 설계하는 방법을 다루었다. Unstructured Perturbation 에 대응하기 좋으며 쉽게 다룰수있는 주파수 응답형 LQR criteria 선정법과, LQR 의 변형으로서 Structured Perturbation 에 대하여 유효한 Performance Criteria Insensitive Control을 제시하고 효과를 살펴보았다.

I. Introduction

One of the essential roles of feedback is to produce a satisfactory control of plants with parameters that are either not known exactly due to modelling errors, or are varying in time during operation. In classical control theory, the robustness of feedback control to parameter uncertainty has been considered by frequency domain methods whereby the gain and phase margin have served to measure the ability of the closed loop system to withstand gain and phase changes in open loop dynamics.

Once it has been shown that the linear optimal quadratic regulators (LQR) satisfy a frequency domain optimality condition (at least 60° of phase margin and infinite gain margin which give a favorite condition of robustness to the closed loop control system) and this optimality in frequency domain can be represented by singular value of return difference matrix (minimun singular value of return difference matrix is equal to or greater than 1), the optimal linear quadratic regulator without or with the Kalman-Bucy filter is approved as true synthesis technique for multivariable system and the singular value of a return difference matrix is used for measuring multivariable feedback control system's robustness.

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Optimal control technique provides explicit solutions for the well defined systems with economic type performance specification. But it has disadvantages as following. /3/

i) It requires access to all the system states
ii) It requires accurate state space model
iii) It is difficult to choose a performance specification
iv) It offers no means of providing dynamic compensator
v) It is difficult to tune up controllers on-line

Much recent work in the field of optimal control theory has been directed towards overcoming these disadvsntages.

For the disadvantage i) observers or Kalman-Bucy fillters are used to reconstitute the inaccessible states for the optimal control feedback. Generally the linear optimal control systems using observers for reconstitution of states lose the LQR's optimality in frequency domain except for the minimum phase system for which one can maintain LQR's robustness by using very fast response Kalman-Bucy filter or by choosing the weighting matrix for input in the quadratic criteria very small for the LQR problem. This result for minimum phase system leads us to obtain dynamic compensator without losing the LQR's optimality in frequency domain, although the procedure is complicated. /5/ /7/ /8/ /17/ /18/

Accuracy of a model is another problem. Here we mention the problem after choosing a definitive model in accurate form. For the disadvantage ii) considerable efforts have been expended for the insensitivity of a feedback control system to parameter variations by choosing the worst case model, by averaging the performance cost for the different possible cases, or by adding noise or uncertainty weighting etc. based upon the sufficient informations of a system. /14/

In the case of the third disadvantage, the problem is delicate in origin. Therefore there is no universal agreement on precisely how the weighting matrix are to be selected for any given application.

The main objective of this paper is to show that the bandwidth and the static gain of a LQR system can be adjusted such as classical dynamic componsator by proper selection of the performance criteria. The possibility of manipulating frequency response for LQR system provides the way of systematic approach for determining the performance criteria. It makes the LQR's optimality in frequency domain to be more useful for the robustness against the unstructured perturbations.

To make LQR system robust against structured perturbations (the tendency of perturbations is known a priori), a method to obtain the control which gives insensitivity for the parameter variations is examined.

2. Selection of LQR performance criteria by frequency response specification

In classical control for a closed loop control system, one looks for the gain change. But if the simple adjustment of the gain is proved not to be sufficient, one searches a dynamic compensator which can modify the configuration of the frecuency response. The LQR type control is based on time domain nature, but often the desired performance of a control system can be expressed by frequency response. /11/ /12/ And some recent results of singular value analysis for robustness can be easily interpreted in frequency response. /6/ /8/ /10/

Let us suppose that we know the frequency response of a system to be controlled $G(j\omega)$ and that we desire a certain controller.

$$\hat{K}(j\omega) = \lambda f(j\omega) \tag{1}$$

Where λ is a pure static gain and $f(j\omega)$ is a frequency operator. We now want to find a LQR controller $K(j\omega)$ which behaves approximately as $\hat{K}(j\omega)$, therefore we can deduce a minimization problem.

minimize

$$\int_{-\infty}^{\infty} [u(j\omega) - \hat{K}(j\omega)\mathcal{E}(j\omega)]^* [u(j\omega) - \hat{K}(j\omega)\mathcal{E}(j\omega)]\, d\omega \tag{2}$$

where $u(j\omega)$ is a process control input produced by a LQR controller $K(j\omega)$,
$\hat{K}(j\omega)$ is a desired controller,
$\mathcal{E}(j\omega)$ is a difference between the desired reference variable $r(j\omega)$ and the output variable $y(j\omega)$

and * means the matrix's complex conjugate transpose.

We assume that it exists a LQR command which satisfies

$$\int_{-\infty}^{\infty} [\hat{K}(j\omega) - K(j\omega)]^* [\hat{K}(j\omega) - K(j\omega)]\, d\omega < \int_{-\infty}^{\infty} \hat{K}(j\omega)^* \hat{K}(j\omega)\, d\omega \text{ or } \int_{-\infty}^{\infty} K(j\omega)^* K(j\omega)\, d\omega \tag{3}$$

then we can derive this inequality from the equation (2)

$$\int_{-\infty}^{\infty} [u(j\omega) - \hat{K}(j\omega)\mathcal{E}(j\omega)]^* [u(j\omega) - \hat{K}(j\omega)\mathcal{E}(j\omega)]\, d\omega \leq \int_{-\infty}^{\infty} [\mathcal{E}^*(j\omega) \hat{K}^*(j\omega) \hat{K}(j\omega) \mathcal{E}(j\omega) + u(j\omega)^* u(j\omega)]\, d\omega \tag{4}$$

where the quantity of the equation (5) must be positive

$$\int_{-\infty}^{\infty} [\hat{K}(j\omega)K(j\omega) + \hat{K}(j\omega)\, K^*(j\omega)\,]\, d\omega \tag{5}$$

The right hand side of the equation (4) can be expressed in time domain by the theory of Parseval.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}[\,\varepsilon^*(j\omega)\hat{K}^*(j\omega)\hat{K}(j\omega)\varepsilon(j\omega)+u(j\omega)^* u(j\omega)\,]\,d\omega$$
$$= \int_0^{\infty}[\,\varepsilon(t)^T W_x\, \varepsilon(t) + u(t)^T u(t)\,]\,dt \tag{6}$$

with

$$W_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{K}(j\omega)^* \hat{K}(j\omega)\, e^{j\omega t}\, d\omega$$
$$= \frac{\lambda^2}{2\pi}\int_{-\infty}^{\infty} f(j\omega)^* f(j\omega)\, e^{j\omega t}\, d\omega$$

It shows that one can adjust the frequency response of a LQR control system by modifying the weighting matrix of the LQR performance criteria. Obviously the controller obtained by such way remains to be LQR controller. Experiences show that this method is very effective for adjustment the bandwidth and modifying the frequency response configuration within the bandwidth. The following example illustrates its simplicity and effectiveness.

example 1

Consider this simple system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -0.01 & -0.001 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{7}$$

$$y(t) = [\,1 \quad 0.1\,]\, x(t) \tag{8}$$

$$J = \frac{1}{2}\int_0^{\infty}\left(x^T \begin{bmatrix} 1 & 0.1 \\ 0.1 & 0.01 \end{bmatrix} x + u^T u \right) dt \tag{9}$$

The LQR system with the performance criteria (9) which means $\hat{K}(jw) = 1$, gives 65°65 phase margin and 1.55 rad/sec cut-off frequency. The letter ① in the table[1] and the Bode diagram in the figure 1 shows this case. /17/

Now we augment static gain by taking $\hat{K}(jw) = 500$ in ②, apply differentiator type with time delay $\hat{K}(jw) = 1 + 10jw$ at ③ and $\hat{K}(jw) = 1 + 100jw$ at ④, integrator type $\hat{K}(jw) = \dfrac{1}{1 + 10jw}$ at ⑤ and double integrator type $\hat{K}(jw) = \dfrac{1}{(1 + 10jw)^2}$ at ⑥.

The results in the table[1] and figure 1 demonstrate that this method is very powerful for manipulations of frequency response in LQR system.

table [1]

	$\hat{K}(jw)$	W_x, (W_u =1)	phase margin	cut-off frequency (rad/sec)
①	1	$W_x = \begin{bmatrix} 1 & 0.1 \\ 0.1 & 0.01 \end{bmatrix}$	65°65	1.55
②	500	$W_x = \begin{bmatrix} 500 & 05 \\ 50 & 5 \end{bmatrix}$	67°32	7.58
③	1+10jw	$W_x = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix}$	89°46	10.23
④	1+100jw	$W_x = \begin{bmatrix} 1 & 100 \\ 100 & 10000 \end{bmatrix}$	90°	100
⑤	$\dfrac{1}{1+10jw}$	$W_x = \begin{bmatrix} 000 \\ 000 \\ 001 \end{bmatrix}$	60°99	0.85
⑥	$\dfrac{1}{(1+10jw)^2}$	$W_x = \begin{bmatrix} 0000 \\ 0000 \\ 0000 \\ 0001 \end{bmatrix}$	60°12	0.631

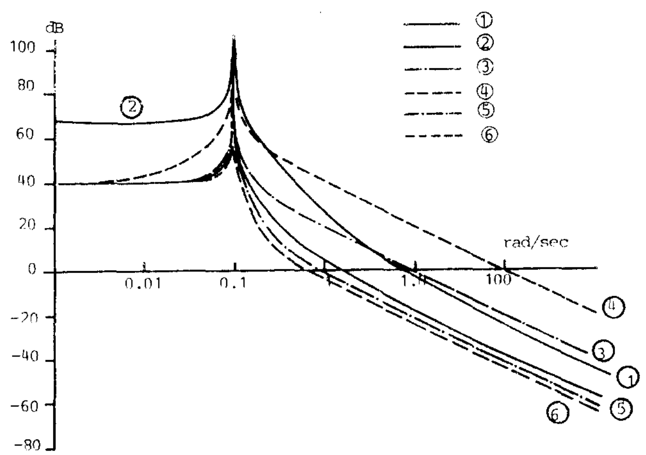

Fig 1 Bode diagram of the LQRs with frequency response specification criteria

3. Performance criteria insensitive control

System uncertainty behavior in low frequency domain can be charaterized by parameter variation in the model of a system. The direct control method introduced in the reference /14/ tries to minimize the expected performance criteria for the probabilistic parameter system model. But the performance criteria insensitive control method treated in this section minimizes the sum of the criteria and its variation which is calculated by the criteria's gradient and the weighting coefficients.

For a plant expressed as the equation (10), We have a ordinary LQR criteria J_0 given by the eq. (11), and its variation δJ_0 where $\delta J_0(F)^2$ means the variations' square with weighting coefficients for the system parameter matrix F and $\delta J_0(B)^2$ is for the matrix B.

$$\dot{x} = F x + B u \tag{10}$$

$$J_0 = \frac{1}{2}\int_0^\infty (x^T W_{x_0} x + u^T W_{u_0} u)\, dt \tag{11}$$

$$\delta J_0 = \delta J_0(F)^2 + \delta J_0(B)^2$$

We assume the control input u(t) to be produced by $u(t) = -C\,x(t)$ where C is the constant feedback gain matrix. Then we have

$$\dot{x}(t) = (F - BC)\, x(t) \tag{12}$$

$$J_0 = \frac{1}{2}\int_0^\infty x^T (W_{x_0} + C^T W_{u_0} C) x \cdot dt \tag{13}$$

The performance criteria J_0 can be expressed by the steady-state Lyapunov matrix S_1.

$$J_0 = \frac{1}{2}\, tr [S_1 X_0] \tag{14}$$

where $X_0 = x_0 x_0^T$ is the initial condition covariance matrix for which we take identy matrix and the positive symmetric matrix S_1 satisfies the Lyapunov relation.

$$(F-BC)^T S_1 + S_1 (F - BC) + W_{x_0} + C^T W_{u_0} C = 0 \tag{15}$$

Then the optimun control problem which optimizes the criteria in the equation (14) is to minimize this following Hamiltonian.

$$H = \frac{1}{2} tr [S_1 + S_2 (F-BC)^T S_1 + S_2 S_1 (F-BC) + S_2 W_{x_0} + S_2 C^T W_{u_0} C] \tag{16}$$

, where S_2 is lagrangian multiplier matrix.
The necessary conditions for optimality to the Hamiltonian are obtained such as

$$\frac{\partial H}{\partial C} = W_{u_0} C S_2 - B^T S_1 S_2 = 0 \tag{17}$$

$$\frac{\partial H}{\partial S_2} = W_{x_0} + C^T W_{u_0} C + (F-BC)^T S_1 + S_1 (F-BC) = 0 \tag{18}$$

$$\frac{\partial H}{\partial S_1} = I + (F-BC) S_2 + S_2 (F-BC)^T = 0 \tag{19}$$

Now suppose that the system parameter matrix F and B vary continuously in the vicinity of its nominal value. The gradients of J_0 with respect to the matrix F and B when the optimal command is applied are obtained as

$$\left[\frac{\partial J_0}{\partial F}\right]_{C=\hat{C}} = \hat{S}_1 \hat{S}_2 \tag{20}$$

$$\left[\frac{\partial J_0}{\partial B}\right]_{C=\hat{C}} = -\hat{S}_1 \hat{S}_2 \hat{C}^T \tag{21}$$

,where $\wedge$ signifies that these matrix variables satisfy the necessary conditions (17) (18) and (19)

Let's define the variation's square of the criteria J_0 , $\delta J_0(F)^2$ and $\delta J_0(B)^2$, by

$$\delta J_0(F)^2 \triangleq tr \sum_{i,j} \delta_{ii}\left(\frac{\partial J_0}{\partial F}\right)\delta_{jj} \left[\delta_{ii}\left(\frac{\partial J_0}{\partial F}\right)\delta_{jj}\right]^T \eta_{ij} \tag{22}$$

$$\delta J_0(B)^2 \triangleq tr \sum_{i,j} \delta_{ii}\left(\frac{\partial J_0}{\partial B}\right)\delta_{jj} \left[\delta_{ii}\left(\frac{\partial J_0}{\partial B}\right)\delta_{jj}\right]^T \xi_{ij} \tag{23}$$

where η_{ij} , ξ_{ij} are weighting coefficents for parameter variation tendency and δ_{ii} , δ_{jj} are square matrix for which all the elements are zero except the element ii or jj which is equal to 1

Let's define now the new criteria J by the sum of J_0 and its variation δJ_0 , then the new Hamiltonian is expressed by

$$\begin{aligned} H = \frac{1}{2} tr\{ & S_1 + \sum_{i,j} \delta_{ii} S_1 S_2 \delta_{jj} [\delta_{ii} S_1 S_2 \delta_{jj}]^T \eta_{ij} \\ & + \sum_{i,j} \delta_{ii} S_1 S_2 C^T \delta_{jj} [\delta_{ii} S_1 S_2 C^T \delta_{jj}]^T \xi_{ij} \\ & + K_2 [W_{x_0} + C^T W_{u_0} C + S_1 (F-BC) + (F-BC)^T S_1] \\ & + K_1 [I + (F-BC) S_2 + S_2 (F-BC)^T] \} \end{aligned} \tag{24}$$

, where K_1 and K_2 are symmetric multiplier matrix.

We will examine this performance criteria insensitive control method to the following simple system. system.

example 2

$$\dot{x} = \begin{bmatrix} 1 & 4 \\ -2 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad (25)$$

$$J_0 = \frac{1}{2}\int_0^\infty \left\{ x^T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x + u^2 \right\} dt \qquad (26)$$

For this system we consider three cases as following

1. To minimize only the criteria J_0 without considering its variation *i.e.* ordinary LQR problem
2. To minimize the sum of J_0 and its variation with the weighting coefficients
$$\eta_{ij} = \begin{bmatrix} 0 & 0 \\ 0.1 & 4 \end{bmatrix} \qquad \xi_{ij} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
3. To minimize the sum of J_0 and its variation with
$$\eta_{ij} = \begin{bmatrix} 0 & 0 \\ 8 & 0.2 \end{bmatrix} \qquad \xi_{ij} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The table [2] shows the closed loop charateristic roots and the system's sensitivity with the relative gradient of the original criteria J_0 with respect to the matrix F and B's variation, when the command minimizing the equation (24) is applied.

Table [2] System's sensitivity and closed loop charateristic roots.

case	coef. η_{ij}	$\frac{1}{J_0}\frac{\partial J_0}{\partial F}$	$\frac{1}{J_0}\frac{\partial J_0}{\partial B}$	λ_i
1	0 0 0 0	0.527 -0.098 -0.103 0.438	-0.266 -1.788	-1.19 $\pm 3.05j$
2	0 0 0.14	0.597 -0.189 -0.080 0.192	0.537 -1.191	-2.36 $\pm 2.46j$
3	0 0 80.2	0.722 -0.156 -0.042 0.347	0.305 -1.531	-1.24 $\pm 2.27j$

It is observed that the sensitivity according to F_{21} and F_{22} 's variation in the cases (2), (3) is diminished in comparison to the case (1).

$$\frac{1}{J_0}\frac{\partial J_0}{\partial F_{22}} = 0.1923 \text{ (case 2)} \Big/ 0.4375 \text{ (case 1)} \qquad (27)$$

$$\frac{1}{J_0}\frac{\partial J_0}{\partial F_{21}} = -0.042 \text{ (case 3)} \Big/ -0.1032 \text{ (case 1)} \qquad (28)$$

The command which minimizes the sum of original criteria and its variations provides system's low sensitivity to the considered parameter's variation. But it is possible that this command can't guarantee obtaining maximum value 1 for minimum singular value of the return difference matrix of the nominal system, when the minimization of criteria's variation is emphasized very much.

4. Conclusions

We have attempted to improve the LQR control method in considering the robustness and the insensitivity.

The LQR control guarantees the maximum stability margin systematically near cut-off frequency for the nominal system, even for a multivariable system. But the LQR has some inconveniences as mentioned in the introduction. The performance criteria selection method based upon frequency response specification is able to resolve the difficulty of criteria selection, which is one of the greatest inconveniences, and provides much flexibility to the LQR system. Such advantage conducts the LQR system to satisfy the diverse requirements for robust stabilization.

We have investigated the performance criteria insensitive control which is a modification of the LQR control under the assumption that only the variation tendency of system parameters is known a priori. It makes the system lose the frequency domain optimality that the LQR system possesses and produces another difficulty to choose the weighting coefficient for the performance criteria's variation. But the performance criteria insensitive control system is very effective in the the view of the insensitivity against the system's variation.

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