선형계통의 파라미터 추정을 위한 최적 입력의 설계

DESIGN OF THE OPTIMAL INPUTS FOR PARAMETER ESTIMATION IN LINEAR DYNAMIC SYSTEMS.

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1. Introduction

It is well known that the parameter estimation accuracy is dependent upon the choice of input signal.

Optimal input means that maximum information about the system can be extracted from the measured input-output data. An optimal input will, in general, depend on the true system except the special case [1] .

To overcome this difficulty we have to do a preliminary experiment to get a first model and assume that a nominal value of the parameter is given. It will be considered as a true value for computing the optimal input. Using the optimal input, an improved model is then estimated.

The problem of designing optimal input for parameter estimation in dynamic systems has been studied by several authors. Most of them imposed constraints on the input signal and in many cases analytic solutions could be obtained [2, 3,4].

Input design with constrained output has also been considered [4,5,6]. Especially Ng., Goodwine and Payne [5] showed that a minimum variance control law with white pertubation signals achieves D-optimality.

Using a Chebyshev system approach Zarrop [4] showed that under a certain condition, D-optimal design could be achieved with finite number of input frequencies without feedback. Recently it is proved by Stoica and Söderström [7] that the optimal input signal can be realized as a certain ARMA process.

This paper considers the problem of optimal input design for the linear regression model with output constraint.

2. Model description

Consider the linear single-input single-output descrete-time systems described by :

$$A(q^{-1})\, y_t = B(q^{-1})\, U_t + e_t \qquad (1)$$

Where $\{e_t\}$ is a zero mean, unit variance white Gaussian noise sequence and q^{-1} is the unit backward shift operator.

The polynomials $A(q^{-1})$ and $B(q^{-1})$ are given as

$$A(q^{-1}) = 1 + a_1 q^{-1} + \dots\dots + a_n q^{-n} \qquad (2)$$
$$B(q^{-1}) = b_1 q^{-1} + \dots\dots\dots\dots + b_m q^{-m}$$

The following assumptions are made.

(A1) The polynomials A and B are relatively prime and the integers n,m are known.

(A2) The polynomial A(z) has all its zeros outside the unit circle.

(A3) The input signal $\{U_t\}$ is a stationary ergodic process, uncorrelated with $\{e_s\}$ for any t and s and is persistently exciting of an appropriate order.

(A4) Asymptotically unbiased and efficient estimator is used.

(A5) A priori estimate θ_0 for the parameter θ is given by means of a preliminary experiment.

The vector θ of unknown parameters to be estimated is

$$\theta = [\, a_1 \,\dots\dots\, a_n \;\, b_1 \,\dots\dots\, b_m \,]^T$$

3. Optimal design procedure

A general measure of the estimation accuracy is given by the covariance matrix of the parameter estimates and (A4) tells that the asymptotic covariance matrix is equal to the inverse of Fisher information matrix M defined by

$$M \triangleq E_{Y|\theta} \left(\frac{\partial L}{\partial \theta}\right)\left(\frac{\partial L}{\partial \theta}\right)^T \qquad (3)$$

Where L is the log-likelihood function log $p(Y|\theta)$ and $\frac{\partial L}{\partial \theta}$ denotes a column vector with i-th component of θ .

An expression for M_θ is derived in detail. It can be shown that L is given by

$$L = -\tfrac{1}{2} \sum_{t=1}^{N} \varepsilon_t^2 + \text{constant} \qquad (4)$$

Where N is the input-output data record length and $\{\varepsilon_t\}$ is the residual sequence defined by

$$\varepsilon_t = A(z^{-1})Y_t - B(z^{-1})U_t \qquad (5)$$

From eq. (4)

$$\frac{\partial L}{\partial \theta} = -\sum_{t=1}^{N} \varepsilon_t \cdot \frac{\partial \varepsilon_t}{\partial \theta} \qquad (6)$$

From eq. (5)

$$\frac{\partial \varepsilon_t}{\partial a_i} = \frac{B(z^{-1})}{A(z^{-1})} \cdot z^{-i} U_t + \frac{1}{A(z^{-1})} z^{-i} e_t$$

$$(i=1,\ldots,n) \qquad (7)$$

$$\frac{\partial \varepsilon_t}{\partial b_i} = -z^{-i} U_t \qquad (i=1,\ldots,m) \qquad (8)$$

Note that $\frac{\partial \varepsilon_t}{\partial \theta_i}$ are statistically independent of ε_t for all t and is influened by the choice of input signals.

Thus

$$M_\theta = \sum_{t=1}^{N} \left(\frac{\partial \varepsilon_t}{\partial \theta}\right)\left(\frac{\partial \varepsilon_t}{\partial \theta}\right)^T \qquad (9)$$

For large N, it is reasonable to consider the average information matrix per sample defined by

$$\bar{M}_\theta = \lim_{N\to\infty} \frac{1}{N} M_\theta$$

$$= E\left(\frac{\partial \varepsilon_t}{\partial \theta}\right)\left(\frac{\partial \varepsilon_t}{\partial \theta}\right)^T \qquad (10)$$

Substituting eq. (7) and (8) into (10) yields

$$M \triangleq M_\theta = E[\varphi(t)\ \varphi^T(t)] + K \cdot \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

$$= M' + M'' \qquad (11)$$

Where $\varphi(t) = [\frac{B}{A} U_{t-1} \cdots \frac{B}{A} U_{t-n}, -U_{t-1} \cdots -U_{t-m}]^T$

$K = E(\frac{1}{A} e_t)^2$

I : n x n identity matrix

Using the Sylvester matrix, the following expression for the first term of eq. (11) can be obtained

$$M' = \sigma^2 \cdot S(B, -A)\ E[\bar{\varphi}(t)\, \bar{\varphi}^T(t)] \cdot S^T(B, -A) \qquad (12)$$

Where $\bar{\varphi}(t) = [\bar{U}_{t-1} \ldots \bar{U}_{t-n} \ldots\ldots \bar{U}_{t-n-m}]^T$

$\sigma^2 = E(\frac{1}{A} U_t)^2$

$$\bar{U}_t = \frac{1}{\sigma A} U_t \qquad (13)$$

$$S(B, -A) = \left[\begin{array}{cccccc} 0 & b_1 & b_2 & \cdots & b_m & 0 \\ 0 & & 0 & b_1 & \cdots & b_m \\ \hline -1 & -a_1 & -a_2 & \cdots & a_n & 0 \\ 0 & & -1 & -a_1 & \cdots & -a_n \end{array}\right] \begin{array}{l} n \text{ rows} \\ \\ m \text{ rows} \end{array}$$

Note that $E[\bar{U}_t^2] = 1$ and M can be determined by P (= n + m − 1) auto-correlation functions of $\bar{U}_t$,

$$\rho_i = E[\bar{U}_t \bar{U}_{t-i}] \qquad i=1,\ldots P \qquad (14)$$

To formulate the optimization problem, we must introduce the output constraint D.

$$D = \{ U_t \mid E[\tfrac{B}{A} U_t]^2 = C \qquad (15)$$

$$= \{ U_t \mid \sigma^2 E[B\bar{U}_t]^2 = C \qquad (16)$$

and a suitable scalar function of M, det M. The use of tr [WM] leads to a tractable quadratic optimization problem but the resulting optimal input may not be persistently exciting.

In view of eq. (11) − (16), The optimal input design is to determine the $\{\rho_i\}$ that maximize the det M subject to D.

It is necessary that R_p ($P = n + m - 1$) be positive definite. Otherwise the system is not identifiable

$$R_k = \begin{bmatrix} 1 & \rho_1 & \cdots & \rho_k \\ \rho_1 & 1 & \ddots & \rho_{k-1} \\ \vdots & & \ddots & \vdots \\ \rho_k & \rho_{k-1} & \cdots & 1 \end{bmatrix} \qquad (17)$$

This constraint can be efficiently tested by the Levinson-Durbin algorithm and the following Lemma [9].

LDA Algorithm

$$\phi_{k+1} = -a_{k+1,k+1} = (\rho_{k+1} + a_{k,1}\rho_k + \cdots + a_{k,k}\rho_1)/\lambda_k^2$$

$$a_{k+1,i} = a_{k,i} - \phi_{k+1}\, a_{k,k+1-i} \quad (i=1,\cdots k)$$

$$\lambda_{k+1}^2 = \lambda_k^2\,(1 - \phi_{k+1}^2) \qquad (18)$$

with initial values

$$\phi_1 = -a_{1,1} = \rho_1 \qquad \lambda_1^2 = 1 - \phi_1^2$$

Lemma

The following two statements are equivalent

i) $|\phi_k| < 1$ $(k=1, \cdots\cdots P)$

ii) Rp is positive definite

Because of identifiability condition, $|\phi_k| = 1$ $(k \leqslant p)$ cannot be allowed for the linear regression model.

In case of $|\phi_k| < 1$ $(k=1, \cdots P)$, eq. (18) yields the following autoregressive process $\bar{U}_t$ of order p.

$$Ap\,(q^{-1})\,\bar{U}_t = W_t$$
$$Ap\,(q^{-1}) = 1 + {}_{p,1}\, q^{-1} + \cdots + {}_{p,p}\, q^{-p} \qquad (19)$$

Where $\{W_t\}$ is a white noise sequence with $E[W_t^2] = {}^2_p$

Note that the AR process $\bar{U}_t$ generated by eq. (19) matches the given $\{\rho_i\}$ $(i=1,\cdots p)$ and the polynomial Ap(Z) has all its zeroes outside the unit disc [9].

Combining eq. (19) with eq. (13), optimal input signal can be realized as an ARMA process.

$$Ap(q^{-1})U_t = \cdot A(q^{-1})\, w_t \qquad (20)$$
$$\text{with } E[w_t^2] = \lambda_p^2$$

4. Example and simulation results

Consider the following example

$$(1 + a_1 q^{-1} + a_2 q^{-2})Y_t = b_1 q^{-1} U_t + e_t$$

with $a_1 = -0.5$ $\quad a_2 = 0.1$ $\quad b = 0.5$

$E[e_t^2] = 1$

The number of Data N=200

Constrained output variance in eq. (16)

C = 5

The following nominal values of parameters are used for optimal input design

$a_1 = -0.5028$

$a_2 = 0.0970$

$b = 0.4972$

We obtain

$a_{p,1} = 0.0890$ $\quad a_{p,2} = -0.01$

$\lambda_p^2 = 0.992$

To compare the estimation accuracy with other inputs, computer simulations has been done for 30 different realizations,

Case I : White input

Case II : deterministic input With finite number of frequencies [6].

Case III : ARMA input

The simulation results are shown in Table 1. Fig 1,2,3 show the mean normalized errors computed from 30 realizations.

5. Conclusion

Optimal input design problem for linear regression model with constrained output variance has been considered.

It is shown that the optimal input signal for the linear regression model can also be realized as an ARMA process.

Monte-Carlo simulation results show that the optimal stochastic input leads to comparatively better estimation accuracy than white input signal.

References

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parameters	a_1	a_2	b
true values	-0.5	0.1	0.5
case I	-0.5028 ±0.0372	0.0964 ±0.0331	0.4968 ±0.0206
case II	0.5036 ±0.0146	0.1031 ±0.0120	0.5010 ±0.0081
case III	-0.4982 ±0.0359	0.1035 ±0.0253	0.5005 ±0.0132

Table. 1 Estimated parameters (mean and standard deviations computed from 30 realizations)

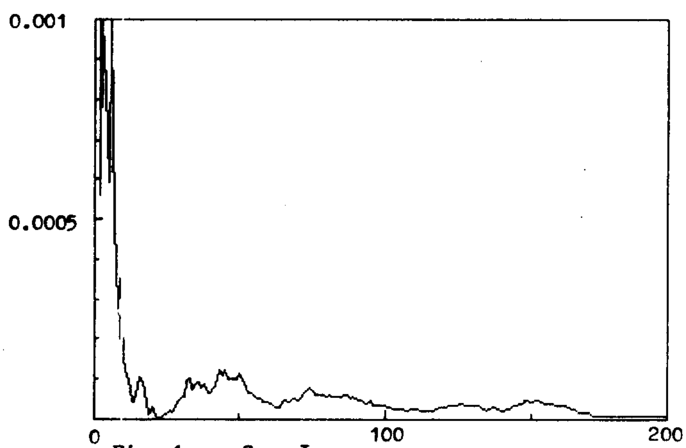

Fig. 1 Case I

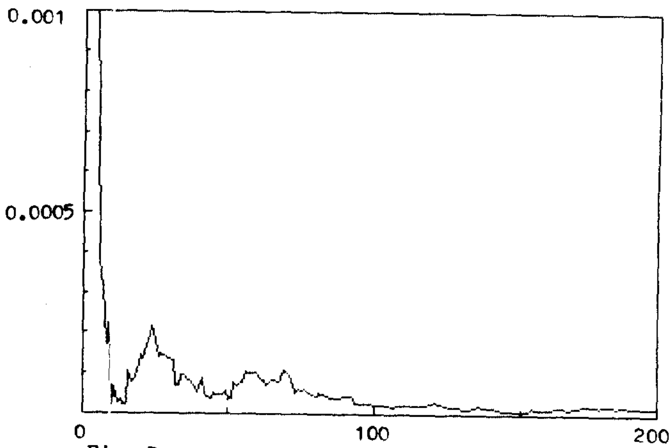

Fig. 3 Case II

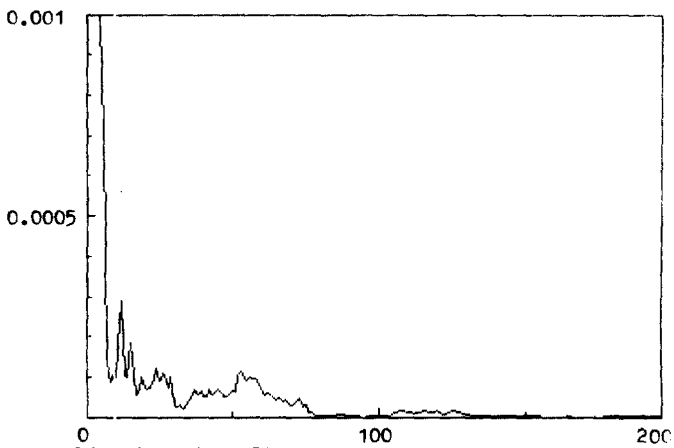

Fig. 2 Case III