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# 행렬 부호함수를 이용한 대규모 선형 시불변 계통의 준최적화

# Near optimal regulator problem for the large-scale linear time-invariant system via matrix sign function

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## 1. Introduction

For a large-scale dynamical systems, mathematical complexities and computational difficulties are encountered when we try to apply them in design of optimal regulator. To solve these, near optimality of large-scale system has been investigated[1] by simplifying original system's model into lower dimensional model through aggregation[1] or singular perturbation method.[1],[4] But determination of aggregation matrix or permutation, rescaling must be carried out in prior to apply these method.

In this paper another approch to near optimal regulator problem is introduced by using matrix sign function. Given large-scale system can be decomposed in its fast and slow subsystems and neglecting its fast subsystem, near optimal regulator can be designed with only its slow subsystem.

## 2. Block-decomposition via matrix sign function

To develop the method for block-decomposition of large-scale system, it is required to review the property of sign function as in the following definition.[3]

Definition 1.

The scalar sign function of a complex variable $\lambda$ with $Re(\lambda)\neq 0$ is defined by

$$sign(\lambda) = \begin{cases} +1 & \text{when } Re(\lambda) > 0 \\ -1 & \text{when } Re(\lambda) < 0 \end{cases}$$

From definition 1 we can extend it to matrix sign function as follows:

Let M be the modal matrix of A(nxn) and

$$J = M^{-1}AM = \text{block diag}[J_+, J_-] \quad (1)$$

where $J_+ \in C^{n_1 \times n_1}$ and $J_- \in C^{n_2 \times n_2}$, with $n_1 + n_2 = n$, are the collection of Jordan blocks associated with the spectrum $\sigma(A) \subset C^+$ and $\sigma(A) \subset C^-$, respectively, where $C^+$ and $C^-$ are the open right and left planes of C, respectively. Then we can write the matrix sign function as follows:

$$\mathrm{sign}(A) = M\ [\mathrm{sign}(J_+) \oplus \mathrm{sign}(J_-)]\ M^{-1}$$
$$= M\ [I_{n1} \oplus (-I_{n2})]\ M^{-1} \qquad (2)$$

And from the definition of the matrix sign function using Riesz projector[3], we can write:

$$\mathrm{sign}(A) = 2\mathrm{sign}^+(A) - I_n$$
$$= I_n - 2\mathrm{sign}^-(A)$$

or
$$\mathrm{sign}^+(A) = \tfrac{1}{2}[I_n + \mathrm{sign}(A)]$$
$$\mathrm{sign}^-(A) = \tfrac{1}{2}[I_n - \mathrm{sign}(A)] \qquad (3)$$

The computaion of sign(A) is possible by the algorithm proposed by Robert[2] and Shieh.[3]

Then the block diagonalization of a sysem map can be accomplished via the matrix sign function as follows:

Let us rewrite Eq (2) as,

$$\mathrm{sign}(A) = M[I_{n1} \oplus (-I_{n2})]\ W$$
$$\triangleq \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} [I_{n1} \oplus (-I_{n2})] \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

$$\text{where } W \triangleq M^{-1} \triangleq \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \qquad (4)$$

and from Eq (4), Eq (3) becomes

$$\mathrm{sign}^+(A) = M[I_{n1} \oplus 0_{n2}]\ W$$
$$= \begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} \begin{bmatrix} W_{11} & W_{12} \end{bmatrix}$$
$$\triangleq M_1 W_1 \qquad (5)$$

$$\text{and } \mathrm{sign}^-(A) = M[0_{n1} \oplus I_{n2}]\ W$$
$$= \begin{bmatrix} M_{12} \\ M_{22} \end{bmatrix} \begin{bmatrix} W_{21} & W_{22} \end{bmatrix}$$
$$\triangleq M_2 W_2 \qquad (6)$$

Let matrices $S_1$ and $S_2$ be defined as

$$S_1 \triangleq \mathrm{ind}\,[\mathrm{sign}^+(A)] = [s_1, s_2, \ldots, s_{n1}]$$
$$S_2 \triangleq \mathrm{ind}\,[\mathrm{sign}^-(A)] = [s_1, s_2, \ldots, s_{n2}]$$

where ind(.) indicates independent columns of $\mathrm{sign}^+(A)$ and $\mathrm{sign}^-(A)$, respectively.

These independent column vectors are selected from the $n(>n_i, i=1,2)$ column vectors of $\mathrm{sign}^+(A)$ and $\mathrm{sign}^-(A)$, respectively

These matrices $S_1$ and $S_2$ can be wrtten as follows:

$$S_1 = \mathrm{sign}^+(A)E_1\ , \quad E_1 \triangleq [e_n^{k1}, e_n^{k2}, \ldots, e_n^{kn1}] \in \mathcal{C}^{n\times n_1}$$
$$S_2 = \mathrm{sign}^-(A)E_2\ , \quad E_2 \triangleq [e_n^{k1}, e_n^{k2}, \ldots, e_n^{kn2}] \in \mathcal{C}^{n\times n_2}$$

where $E_1$ and $E_2$ are elementary matrices.

From Eq (5) and Eq (6) we obtain.

$$S_1 = M_1 W_1 E_1 \triangleq M_1 H_1 \quad , \text{ where } H_1 \triangleq W_1 E_1 \qquad (7)$$
$$S_2 = M_2 W_2 E_2 \triangleq M_2 H_2 \quad , \text{ where } H_2 \triangleq W_2 E_2 \qquad (8)$$

According to Sylvester's inequality we obtain $\mathrm{rank}(H_1) = n_1$ and $\mathrm{rank}(H_2) = n_2$ or $H_1$ and $H_2$ are nonsingular.

If we define right block modal matrix $M_r$ as

$$M_r \triangleq [S_1\ \ S_2] = MH \qquad (9)$$

where $H = [H_1 \oplus H_2]$. And from Eq.(1)

$$AM_r = AMH = MJM^{-1}MH$$
$$= MH\left(H^{-1} \begin{bmatrix} J_+ & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & J_- \end{bmatrix} H\right) \triangleq M_r A_R \qquad (10)$$

Hence we can write

$$A_R = M_r^{-1} A M_r$$

We can apply the previous result to block decomposition of a system in the following theorem.

Theorem 1

Let a linear time invariant system be given by

$$\dot{X} = AX + Bu \quad , X \in R^{n\times 1}, \ u \in R^{m\times 1} \qquad (11\text{-a})$$

$$y = CX \qquad , y \in R^{p\times 1} \qquad (11\text{-b})$$

and

$$X = M_r X_R \triangleq M_r [X_s^T \ X_f^T]^T, \ X_s \in R^{n_1\times 1}, \quad X_f \in R^{n_2\times 1} \qquad (12)$$

Then the block decomposed system of Eq.(11-a,b) using Eq.(12) becomes

$$\dot{X}_R = A_R X_R + B_R u \qquad (13\text{-a})$$

$$y = C_R X_R \qquad (13\text{-b})$$

where

$$A_R = M_r^{-1} A M_r = \text{block diag}[A_{R1}, \ A_{R2}] \qquad (14\text{-a})$$

$$B_R = M_r^{-1} B = [B_{R1}^T, \ B_{R2}^T]^T \qquad (14\text{-b})$$

$$C_R = C M_r = [C_{R1}, \ C_{R2}] \qquad (14\text{-c})$$

Proof) From Eq.(4)

$$WM = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \begin{bmatrix} M_1 & M_2 \end{bmatrix} = \begin{bmatrix} W_1 M_1 & W_1 M_2 \\ W_2 M_1 & W_2 M_2 \end{bmatrix} = \begin{bmatrix} I_n & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & I_{n2} \end{bmatrix}$$

and

$$M_r^{-1} M_r = \begin{bmatrix} S_1^+ \\ S_2^+ \end{bmatrix} \begin{bmatrix} S_1 & S_2 \end{bmatrix} = \begin{bmatrix} S_1^+ S_1 & S_1^+ S_2 \\ S_2^+ S_1 & S_2^+ S_2 \end{bmatrix}$$

where '+' denotes pseudo inverse and

$$S_1^+ S_1 = I_{n1}, \ S_2^+ S_2 = I_{n2}, \ S_1^+ S_2 = 0_{n_1\times n_2}, \ S_2^+ S_1 = 0_{n_2\times n_1} \qquad (15)$$

Therefore Eq.(14-a) is

$$M_r^{-1} A M_r = \begin{bmatrix} S_1^+ \\ S_2^+ \end{bmatrix} A \begin{bmatrix} S_1 & S_2 \end{bmatrix} = \begin{bmatrix} S_1^+ A S_1 & S_1^+ A S_2 \\ S_2^+ A S_1 & S_2^+ A S_2 \end{bmatrix} \qquad (16\text{-a})$$

Define

$$A_{R1} \triangleq S_1^+ A S_1 \ , \ S_1 A_{R1} = A S_1 \qquad (16\text{-b})$$

$$A_{R2} \triangleq S_2^+ A S_2 \ , \ S_2 A_{R2} = A S_2 \qquad (16\text{-c})$$

From Eq.(16-b,c), off-diagonal block of Eq.(16-a) can be obtained as follows

$$S_1^+ A S_2 = S_1^+ S_2 A_{R2} = 0_{n_1\times n_2}, \ S_2^+ A S_1 = S_2^+ S_1 A_{R1} = 0_{n_2\times n_1}$$

Eq.(14-b,c) can be proved in the same way.

(Q.E.D.)

From theorem 1 and reference (5) we can obtain the reduced order model of a system with its dominant(slow) modes as follows

$$\dot{X}_s = A_{R1} X_s + B_{R1} u_r \qquad (17\text{-a})$$

$$y = C_{R1} X_s + C_{R2} (-A_{R2})^{-1} B_{R2} u_r \qquad (17\text{-b})$$

$$\cong y_r$$

3. Near optimal regulator problem

To obtain the optimal control law for Eq.(1), we set the performance index J and $J_R$ as follows:

$$J = \tfrac{1}{2}\int_0^\infty (y^T y + u^T R u) dt \qquad (18)$$

$$\cong \tfrac{1}{2}\int_0^\infty (y_r^T y_r + u_r^T R u_r) dt \triangleq J_R \qquad (19)$$

Now the problem is to find a near optimal control law $u_r^*$ to minimize the Eq.(19).

From Eq.(17-b) we can get

$$J_R = \tfrac{1}{2}\int_0^\infty (X_s^T C_{R1}^T C_{R1} X_s + 2u_r^T D^T C_{R1} X_s + u_r^T R_0 u_r) dt \qquad (20)$$

where $D = -C_{R2} A_{R2}^{-1} B_{R2}$, $R_0 = D^T D + R$.

Then we can find the near optimal control

$$u_r^* = -R_0^{-1} (D^T C_{R1} + B_{R1}^T K_s) X_s \qquad (21)$$

where $K_s$ is a solution of Riccati equation

$$0_{n_1\times n_1} = K_s (A_{R1} - B_{R1} R_0^{-1} D^T C_{R1}) + (A_{R1} - B_{R1} R_0^{-1} D^T C_{R1})^T K_s + C_{R1}^T (I_{n1} - D R_0^{-1} D^T) C_{R1} - K_s B_{R1} R_0^{-1} B_{R1}^T K_s$$

From Eq.(12) $X_f$ and $X_s$ was defined by mapping $M_r^{-1}$ as follows

$$\begin{bmatrix} X_s \\ X_f \end{bmatrix} = M_r^{-1} X = \begin{bmatrix} S_1^+ \\ S_2^+ \end{bmatrix} X$$

Hence $X_s$ can be written

$$X_s = S_1^+ X \qquad (22)$$

As a result we can obtain near optimal regulator control $u_r^*$ in Eq.(21) in terms of X as follows

$$u_r^* = -R_0^{-1}(D^T C_{R1} + B_{R1}^T K_s) S_1^+ X$$

4. Numerical Example

Consider a fifth-order linear time-invariant system.

$$\dot{X} = \begin{bmatrix} -2. & -1. & -0.5 & -0.1 & -0.05 \\ 0.5 & -1. & 0.2 & -4. & 0.8 \\ 1. & -0.1 & -0.25 & 0. & -1. \\ 0. & 1. & -0.1 & -0.1 & 0. \\ 1. & -1. & -1. & 0.5 & -5. \end{bmatrix} X + \begin{bmatrix} 1. \\ 0. \\ 1. \\ 1. \\ 1. \end{bmatrix} u$$

$$y = \begin{bmatrix} 1. & 0. & 0. & 0.05 & -0.01 \\ 0. & 1. & 0.01 & 0. & 0. \end{bmatrix} X$$

Let the performance index be $J = \frac{1}{2}\int_0^\infty (y^T y + u^T u)dt$ and $J_R = \frac{1}{2}\int_0^\infty (y_r^T y_r + u_r^T u_r)dt$ for optimal and near optimal regulator, respectively.

As a result of simulations, optimal and near optimal controls are obtained as follows

$$u^* = -0.1479x_1 + 0.11814x_2 - 0.0195x_3 - 0.8565 x_4 + 0.04659x_5$$

$$u_r^* = -0.15386x_1 + 0.1245x_2 - 0.0181x_3 + 0.02427x_5$$

The response curves of $y_{opt}$, $y_r$, and $y_{spt}$ are compared in Fig.1 with their performances. Where $y_{spt}$ denotes the response of singular perturbation method.

5. Conclusions

In this paper near optimal regulator for large scale system has been designed by using lower order model via matrix sign function. As illustrated in the numerical example, proposed near optimal regulator problem has the better response than the one by singular perturbation method and orther near optimal problem. Proposed algorithm can give solution to the difficulties in aggregation and singular perturbation and is useful for the near optimal regulator problem of multivariable systems with widely spaced eigenvalues.

References

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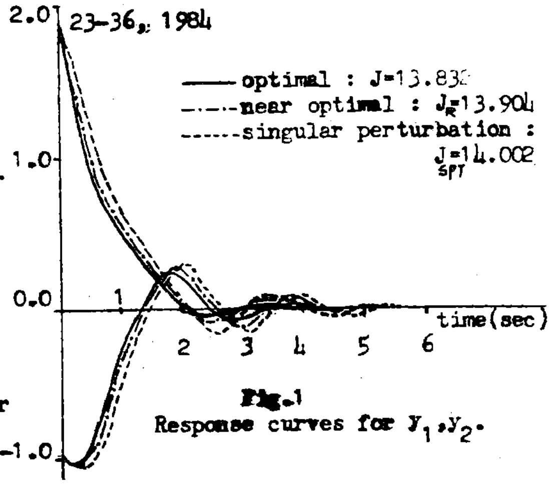


Fig.1
Response curves for $y_1$, $y_2$.