A Study of Stochastic Bilinear Systems as an Economic Modelling.

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1. Introduction

Mohler developed stochastic bilinear systems that are the diffusion models for migration of people, biological cells, etc., [1,2,3,4,5]. In the study and analysis of the behavor of economic systems economic theoretical considerations indicate nonlinear on some degree of nonlinearity in the system. While the variables are defined every where(in time) observation data is available only at discrete time points. Moreover disturbances(noises) is almost always present in the data and in the system itself [6,7]. Because of the mathematical difficulties involved in treating nonlinear on linearized system model. The discrete observations provide the basis for then adopting a discrete time linear system. The noise is in the process and observations is then in corporated usually in an additive fashion, [8,9].

In this paper we consider a class of nonlinear scalar valued stochastic systems of the bilinear type and show that the compromise modelling approach outline above can destroy most of the original system's behaviour.

2. Bilinear Stochastic Differential Equations

Consider the general scalar stochastic differential equation

$$dx_t = f(x_t,t)dt + g(x_t,t)dw_t, \qquad (2\text{-}1)$$
$$x_{t_0} = x_0, \quad t\in[t,t_N], \ t_N<\infty$$

where w_t is a Wiener process. Always assuming that x_t evolves on a regular probability space solutions to (2-1) are defined.

The specific form of (2-1) of interest to us is the bilinear stochastic equation

$$dx_t = (A(t)x_t+a(t))dt + \sum_{i=1}^{m}(B_i(t)x_t+b_i(t)) \cdot dw_t^i, \qquad (2\text{-}2)$$

with x_t scalar valued. This formulation now supplies the flexibility of incorporating more than one exogeneous and/or instrument variable in the model through the variables a, B_i, and b_i, $i=1,2,\ldots,m$. The inhomogeneous process (2-2) has a drift vector $A(t)x_t+a(t)$ and diffusion coefficient $(\sum_{i=1}^{m}B_i(t)x_t+b_i(t))^2$. A solution to the nonhomogeneous process can be espressed as

$$x_t = K(t,t_0)+\int_{t_0}^{t}K(t,\tau)(a(\tau)-\sum_{i=1}^{m}B_i(\tau)b_i(\tau))d\tau + \int_{t_0}^{t}K(t,\tau)\sum_{i=1}^{m}b_i(\tau)dw_\tau^i, \qquad (2\text{-}3)$$

$$K(t,\tau) = \mathrm{Exp}(\int_{t_0}^{t}(A(\tau)-\frac{1}{2}\sum_{i=1}^{m}B_i(\tau))d\tau + \int_{t_0}^{t}\sum_{i=1}^{m}B_i(\tau)dw_\tau^i), \qquad (2\text{-}4)$$

The first term in (2-3) is simply the solution to the homogeneous equation with both the second and third terms stochastic in nature since K is a stochastic process. The second term will have a lognormal, but the last term is further cortupted by the Gaussian element $\sum_{i=1}^{m}b_i(\tau)dw_\tau^i$. Since the sum of lognormally distributed random variables is not lognormally distributed the solution of (2-3) will not have a lognormal distribution even if $b_i(t)=0$, $i=1,2,\ldots,m$. with very little hope of finding a simple probability law for x_t we have to be satisfied with calculating the moments. Taking expectation in (2-2) we obtain the moment equation;

$E\{x_t\} = m(t)$

$$\dot{m}(t) = A(t)m(t) + a(t), \qquad (2\text{-}5)$$

$$m(t_0) = E\{x_0\}$$

and similarly the second moment $P(t)=E\{(x_t-m(t))^2\}$

$$\dot{P}(t) = 2\,A(t)+\frac{1}{2}\sum_{i=1}^{m}B_i^2(t)\;P(t)\;+\sum_{i=1}^{m}(B_i^2(t)\cdot m^2(t)\;+2m(t)B_i(t)b_i(t)\;+\;b_i^2(t)),$$

$$P(t_0)=E\{(x_{t_0}-m(t_0))^2\} \qquad (2\text{-}6)$$

We conclude this section by obtaining a solution to a stochastic bilinear economic model due to D' Alessadro[10]. Consider the following simple capital formation model where

$$\dot{C}(t) = s(t)F(t) - \mu C(t), \quad t\in[0,t_N]$$

$$C(0) = C_0,$$

where μ is the constant rate of capital depreciation, $s(t)$ the total(public and private) saving ratio and $F(t)$ is a linear production function of the form

$$F(t) = \alpha C(t) + \beta L(t), \quad \alpha,\beta>0, \text{ constants.}$$

Assuming that the labor force L grows exogeneously at rate ; i.e.,

$$\dot{L}(t) = \lambda L(t), \quad \lambda>0$$

$$L(0) = L_0.$$

We obtain a solution for the case where the savings ratio is distributed by white noise,

$$s(t) = \bar{s}(t) + n_t$$

n_t is Gaussian with zero mean and unit variance and has the Wiener process w_t associated with it; $\bar{s}(t)$ being the mean saving ratio. Solving for L we obtain $L(t)=L_0\mathrm{Exp}(\lambda t)$ and bilinear in the form,

$$dC_t = ((\alpha\bar{s}(t)-\mu)C_t + L_0\bar{s}(t)e^{\lambda t})dt + (\alpha C_t+\beta L_0 e^{\lambda t})dw_t, \qquad (2\text{-}7)$$

$$C_{t_0} = C_0.$$

Notice that even in this simple case, a linear model will result only if $\alpha=0$, i.e., no capital goods being consumed in production. The kernel of (2-4) is now given by

$$K(t,s) = \mathrm{Exp}(\alpha(w_t-w_s))\mathrm{Exp}(\alpha\int_s^t \bar{s}(\tau)d\tau-(u+\frac{\alpha}{2})^2)(t-s)), \qquad (2\text{-}8)$$

and the solution of (2-3) for this case becomes

$$C_t=K(t,t_0)C_0+\int_0^t K(t,\tau)(\bar{s}(\tau)-\alpha)L_0e^{\lambda\tau}d\tau + \beta L_0\int_0^t K(t,\tau)e^{\lambda\tau}\,dw_\tau\,, \qquad (2\text{-}9)$$

The first term in (2-9) is the solution to the homogeneous equation,

$$dC_t=(\alpha\bar{s}(t)-\mu)C_tdt + \alpha C_tdw_t$$

or the case when $\beta=0$, i.e.,when no labor is used in production on when labor participates with zero efficiency, while the second and third terms add components due to labor involvement. Note further that the term $\mathrm{Exp}(\alpha(w_t-w_s))$ in (2-8), reflecting essentially the accumulative effect of the disturbances, depends only on the ratio of capital goods consumed in production. Moreover, the local disturbances in the process is contained in the last term of (2-9), i.e., $\beta L_0 e^{\lambda t}dw_t$ and depends only on the labor efficiency coefficient β.

Dynamic equations for the first two moments of C_t can be obtained from (2-5) and (2-6) and are given by

$$\dot{m}_c(t)=(\alpha\bar{s}(t)-\mu)m_c(t) + \bar{s}(t)\beta L_0e^{\lambda t}$$

and

$$\dot{P}_c(t)=2(\alpha\bar{s}(t)-u+\frac{\alpha^2}{2})P_c(t)+(\alpha m_c(t)+\beta L_0e^{\lambda t})^2$$

$$=2(\alpha\bar{s}(t)-\mu+\frac{\alpha^2}{2})P_c(t)+\bar{F}(t)^2$$

where $\bar{F}(t) = E[F(t)]$ is the expected production. Since both moment equations are linear with strictly positive solutions. To ensure a bounded variance as $t_N\to\infty$ (necessary to avoid the capital formation from becoming completely random) it is therefore necessary to have

$$\alpha\int_0^t\bar{s}(\tau)d\tau + (\frac{\alpha^2}{2}-\mu)(t-t_0)\leq 0,$$

for large t,

or

$$\frac{1}{t}\int_0^t\bar{s}(\tau)d\tau\leq\frac{1}{\alpha}(\mu-\frac{\alpha^2}{2}).$$

3. Equivalent Discrete Time Systems

In this section the closed form solutions of section 2 are employed to obtain the equivalent discrete time bilinear stochastic systems , i.e., relate the solutions at discrete time points with one another. It is necessary to keep all discretizing arguments consistent with definitions in Ito[11,12,13] and very effort is made to present equations already adapted for computations.

Consider any n-partition of interval $[t_0, t_N)$ with time points t_i, $i=0,1,2,\ldots,N$ where $t_0<t_1<t_2<\ldots<t_N$. By defining

$$\alpha(t_k) = (a(t_k) - B(t_k)b(t_k))(t_{k+1}-t_k)$$

and

$$\rho_{t_{k+1}} = b(t_k)(w_{t_{k+1}} - w_{t_k})$$

where the random variables $(\rho_{t_{k+1}})$ form a sequence of independent Gaussian random variables with zero mean and variance

$$\sigma_\rho^2(t_{k+1}) = b^2(t_k)(t_{k+1}-t_k)$$

For consecutive time steps we obtain the discrete equation,

$$x_{t_q} = x_{t_{q-1}}\Phi(t_q,t_{q-1})r_{t_q} + \Phi(t_q,t_{q-1})(\alpha(t_{q-1}) + \rho_{t_q})r_{t_q}, \qquad (2\text{-}10)$$

where the function

$$\Phi(t_q,t_p)=\mathrm{Exp}\left(\sum_{k=p}^{q-1}\left(A(t_k)-\frac{B^2(t_k)}{2}\right)(t_{k+1}-t_k)\right)$$

act as the usual state transition function. But before the moment processes can be obtained it is necessary to take into account that ρ_{t_q} and r_{t_q} are indeed define over the same forward difference $w_{t_q}-w_{t_{q-1}}$.

By definition,

$$r_{t_q} = \mathrm{Exp}(B(t_{q-1})(w_{t_q}-w_{t_{q-1}})),$$

$$\rho_{t_q} = b(t_{q-1})(w_{t_q}-w_{t_{q-1}})$$

and by keeping $B(t_{q-1})$ and $b(t_{q-1})$ constant, a strictly ordered M partition of $[t_{q-1},t_q]$ reveals by standard arguments that

$$E[r_{t_q}\rho_{t_q}]=B(t_{q-1})b(t_{q-1})(t_q-t_{q-1})=\Delta(t_{q-1})$$

$$E[r_{t_q}^2\rho_{t_q}]=2B(t_{q-1})b(t_{q-1})(t_q-t_{q-1})$$

while

$$E[r_{t_q}^2\rho_{t_q}^2]=b^2(t_{q-1})(t_q-t_{q-1})=\delta(t_{q-1})$$

Using these expressions the mean process is given by

$$m(t_q)=\Phi(t_q,t_{q-1})(m(t_{q-1})+\alpha(t_{q-1}))\mathrm{Exp}\left(-\frac{1}{2}\sigma_\xi^2(t_q)\right)+\Phi(t_q,t_{q-1})\Delta(t_{q-1}),$$

$$m(t_0)=E[x_{t_0}] \qquad (2\text{-}11)$$

while the covariance process is given by

$$\begin{aligned}P(t_q)= {} & \Phi^2(t_q,t_{q-1})P(t_{q-1})\mathrm{Exp}(2\sigma_\xi^2(t_q)) \\ & +\Phi^2(t_q,t_{q-1})(m(t_{q-1})+\alpha(t_q))^2 \\ & \cdot\mathrm{Exp}(\sigma_\xi^2(t_q))[\mathrm{Exp}(\sigma_\xi^2(t_q))-1] \\ & +\Phi^2(t_q,t_{q-1})(m(t_{q-1})+\alpha(t_q))\Delta(t_{q-1})) \\ & (4-\mathrm{Exp}(\sigma_\xi^2(t_q)/2))+\Phi(t_q,t_{q-1}) \\ & (\delta(t_{q-1})-\Delta^2(t_{q-1}))\end{aligned} \qquad (2\text{-}12)$$

where $P(t_0)=E[(x_{t_0}-m(t_0))^2]$.

Note that as in the continuous case the elements of the homogeneous can be easily extracted from the inhomogeneous model and its moment processes.

4. Application

Let us consider a simple growth(decay) model of Harrod-Domar type, where the state was disturbed by some policy or natural disaster at the time instances t_p and again at t_m. Take

$$dx_t = -\lambda x_t dt + B(t)x_t dw_t, \quad t\in[0,T)$$

where is the normal growth constant $\lambda>0$ say and the policy variable

$$B(t) = \begin{cases}1, & t=t_p \text{ and } t=t_m,\ 0<t_p<t_m<T \\ 0, & \text{elsewhere}\end{cases}$$

By selecting an N-partition of $[0,T)$, with t_m and t_p included as sample points and say of equal length $t_{k+1}-t_k=\Delta$ all k. Using the discrete time models

$$x_{t_{k+1}}=\begin{cases}\mathrm{Exp}(-\lambda k)x_{t_0}, & 0\le k<p \\ \mathrm{Exp}(-\lambda k)x_{t_0}r_{t_p}, & p\le k<m \\ \mathrm{Exp}(-\lambda k)x_{t_0}\cdot r_{t_p}\cdot r_{t_m}, & m\le k<N\end{cases}$$

where

$$r_{t_p} = \mathrm{Exp}(B(t_p)(w_{t_{p+1}} - w_{t_p})).$$

With the numbers r_{t_p} and r_{t_m} strictly positive a possible solution is illustrated in figure 1.

Select now a piecewise constant policy variable $a(t)$ and to depict local disturbances a small constant 0 and incorporate them into the model giving

$$dx_t = (-\lambda x_t + a(t))dt + (B(t)x_t + \beta)dw_t$$

Solutions for the discrete time system is given by

$0 \le q < p$

$$x_{t_q} = \mathrm{Exp}(-\lambda q\Delta)x_{t_0} + \sum_{k=0}^{q-1} \mathrm{Exp}(-\lambda k\Delta)(a(t_k) + \xi_{t_k})$$

$p \le q < m$

$$x_{t_q} = \mathrm{Exp}(-\lambda q\Delta)r_{t_p}x_{t_0} + \sum_{k=0}^{p-1} \mathrm{Exp}(-\lambda k\Delta) \cdot (a(t_k) + \xi_{t_k})r_{t_p} + \sum_{k=p}^{q-1} \mathrm{Exo}(-\lambda k\Delta) \cdot (a(t_k) + \xi_{t_k})$$

$m \le q < N$

$$x_{t_q} = \mathrm{Exp}(-\lambda q\Delta)r_{t_p}r_{t_m}x_{t_0} + \sum_{k=0}^{p-1} \mathrm{Exp}(-\lambda k\Delta) \cdot (a(t_k) + \xi_{t_k})r_{t_p}r_{t_m} + \sum_{k=p}^{m-1} \mathrm{Exp}(-\lambda k\Delta) \cdot (a(t_k) + \xi_{t_k})r_{t_m} + \sum_{k=m}^{q-1} \mathrm{Exp}(-\lambda k\Delta) \cdot (a(t_k) + \xi_{t_k})$$

A typical solution is sketched in figure 2.

5. Concluding Remarks

In contrast with mechanical and control systems, dynamic economic systems have the feature that their states remain strictly positive in time. This is an important consideration in the construction of stochastic economic models and is not always assured in linear systems with additive noise. The strict positive feature of the bilinear system or at least the homogeneous case, should be seriously considered as an alternative to linearization in stochastic model construction

Random processes with a lognormal distribution appear frequently in economics, as for example in the distribution of incomes and in the theory of consumer behavior. This is true for birth/death type growth models since they have particularly attractive diffusion process equivalents (1).

The results of section 3 and 4 speak for themselves and essentially give a warning that great care should be taken where treating nonlinear stochastic system in a discrete time form. It is essential that any discretization argument should be based on the closed solution form of the system rather than the model itself.

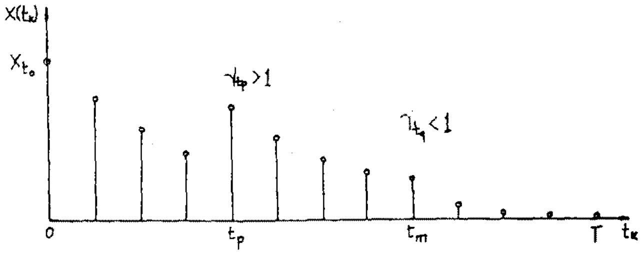

Figure 1.

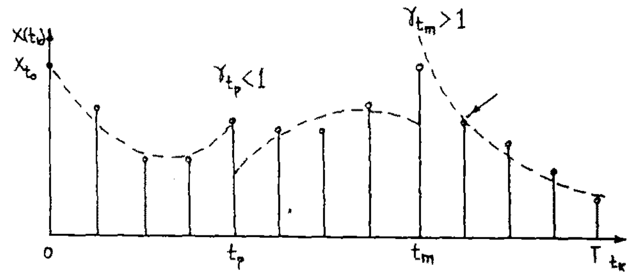

Figure 2.

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