Application of Singular Value Decomposition Method to Image Reconstruction

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Singular value decomposition (SVD) method of solving least square problem is applied to the C.T. (computed tomography) image reconstruction. This method can overcome the ill-conditioning nature of the problems by trading off between noise and signal quality.

Usually C.T. image reconstruction is achieved by combining the filtered back projection and interpolation. It can be also achieved through algebraic image restoration techniques. Using algebraic methods, the imaging system can be modeled as a discretized spatially variant linear system superimposed by additive noise. It can be modeled as,[1]

$$\underline{g} = W\underline{f} + \underline{n} \qquad (1)$$

where $\underline{f}$ is an original image vector (N-vector),
$\underline{n}$ is a noise vector in the sensor (M-vector),
$\underline{g}$ is a projection data vector (M-vector), and
W is a projection matrix (M*N-matrix).

We assume that the modeling is accurate, then the problem is to estimate $\underline{f}$, given $\underline{g}$, $\underline{n}$ and W. A good estimate is [2]

$$\hat{\underline{f}} = W^{+}\underline{g} \qquad (2)$$

where $W^{+} = V S_k^{-1} U^t$ is a pseudo-inverse of W,
$W = U S_k V^t$ is a singular value decomposition of W, and
S_k is an M*N matrix with only ordered K-diagonals.

$\hat{\underline{f}}$ in Eq.(2) is then the minimum norm least square solution to Eq.(1)

when $\underline{n} = \underline{0}$. But in the presence of noise, there occurs an inevitable noise amplification from the ill-conditioning nature of the problems. So we modify S_k^{-1} by setting the smaller singular values to zero, that is to replace S_k^{-1} by S_l^{-1}, where $l < k$.

From Eq.(1),(2) and using matrix outer product expansions, the reconstructed image can be written as follows;

$$\hat{\underline{f}} = \sum_{i=1}^{l} \left[(\underline{v}_i, \underline{f})\ \underline{v}_i + \lambda_i^{-1/2}\ (\underline{u}_i, \underline{n})\ \underline{v}_i \right] \tag{3}$$

where $\underline{u}_i$, $\underline{v}_i$, and $\lambda_i^{1/2}$ are the i-th column vectors of U, V and i-th singular value of W, respectively. Then residual error vector $\underline{r}$ is,

$$\underline{r} = \underline{f} - \hat{\underline{f}} = -\sum_{i=1}^{l} \lambda_i^{-1/2} (\underline{u}_i, \underline{n})\ \underline{v}_i + \sum_{i=l+1}^{N} (\underline{v}_i, \underline{f})\ \underline{v}_i \tag{4}$$

From Eq.(4), we know that by introducing modified SVD, we can reduce noise with some loss of the informations which are the components of $\mathcal{V}$, where $\mathcal{V} = \underline{v}_{l+1} \oplus \underline{v}_{l+2} \oplus \cdots \oplus \underline{v}_k$. If we increase l, the first summation in Eq.(3) will be more closer to original object, but the SNR becomes smaller. Therefore one has to balance between the two effects reasonably. One possibility is to stop at the term where the λ_i is smaller than $\gamma \cdot SNR^{-1}$, where γ is a constant.

The experimental results simulated with computer will be demonstrated.

REFERENCES

1. Kwok-Kai Chan, "Transverse Axial Positron Camera," Ph.D Thesis in UCLA, 1976
2. Lawson and Hanson, Solving Least Squares Problems, Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1974