Abstract. The aim of the present paper is to study 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection. Firstly, we prove that extended generalized $\mathcal{M}$-projective $\phi$-recurrent 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection is an $\eta$-Einstein manifold with respect to Levi-Civita connection under some certain conditions. Later we study some curvature properties of 3-dimensional trans-Sasakian manifold admitting the above connection.

1. Introduction

Let $(M, \phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact metric manifold. Then the product $\bar{M} = M \times \mathbb{R}$ has a natural almost complex structure $J$ with the product metric $G$ being Hermitian metric. The geometry of the almost Hermitian manifold $(\bar{M}, J, G)$ gives the geometry of the almost contact metric manifold $(M, \phi, \xi, \eta, g)$. Sixteen different types of structures on $M$ like Sasakian manifold, Kenmotsu manifold etc are given by the almost Hermitian manifold $(\bar{M}, J, G)$. The notion of trans-Sasakian manifolds were introduced by Oubina [10] in 1985. Then J. C. Marrero [7] has studied the local structure of trans-Sasakian manifolds. In general a trans-Sasakian manifold $(M, \phi, \xi, \eta, g, \alpha, \beta)$ is called a trans-Sasakian manifold of type $(\alpha, \beta)$. Trans-Sasakian manifold of type $(0, 0), (\alpha, 0), (0, \beta)$ are called cosymplectic, $\alpha$-Sasakian and $\beta$-Kenmotsu manifold respectively. Marrero has proved that trans-Sasakian structures are generalized quasi-Sasakian structure. He has also proved that
a trans-Sasakian manifold of dimension \( n \geq 5 \) is either cosymplectic or \( \alpha \)-Sasakian or \( \beta \)-Kenmotsu manifold. So, we have considered here three dimensional trans-Sasakian manifold.

The notion of a semi-symmetric linear connection on a differential manifold has been first introduced by Friedmann and Schouten [4] in 1924. In 1932 Hayden has given the idea of metric connection with torsion on Riemannian manifold in [5]. Yano [15] has given a systematic study of semi-symmetric connection on Riemannian manifold in 1970. Later K. S. Amur and S. S. Pujar [1], C. S. Bagewadi [3], Sharafuddin and Hussain (1976) [13] and others have also studied semi-symmetric connection on Riemannian manifold. S. Pahan, A. Bhattacharyya studied some curvature properties of projective curvature tensor with respect to semi-symmetric connection on a three dimensional trans-Sasakian manifold in [8].

Our aim is to study different types of curvature tensors on 3-dimensional tran-Sasakian manifold and their properties under certain condition with respect to semi symmetric metric connection.

2. Preliminaries

An \( n (=2m+1) \) dimensional Riemannian manifold \((M, g)\) is called an almost contact manifold if there exists a \((1,1)\) tensor field \( \phi \), a vector field \( \xi \) and a 1-form \( \eta \) on \( M \) such that

\[
\phi^2(X) = -X + \eta(X)\xi,
\]

\[
\eta(\xi) = 1, \eta(\phi X) = 0,
\]

\[
\phi \xi = 0,
\]

\[
\eta(X) = g(X, \xi),
\]

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),
\]

\[
g(X, \phi Y) + g(Y, \phi X) = 0,
\]

for any vector fields \( X, Y \) on \( M \). An odd dimensional almost contact metric manifold \( M \) is called a trans-Sasakian manifold if it satisfies the following condition

\[
(\nabla_X \phi)(Y) = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\},
\]
for some smooth functions $\alpha, \beta$ on $M$ and we say that the trans-Sasakian structure is of type $(\alpha, \beta)$. For an $n$-dimensional trans-Sasakian manifold \cite{9}, from (2.7) we have

\begin{align}
(2.8) \quad \nabla_X \xi &= -\alpha \phi X + \beta (X - \eta(X)\xi), \\
(2.9) \quad (\nabla_X \eta)(Y) &= -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y).
\end{align}

In an $n$-dimensional trans-Sasakian manifold, we have

\begin{align}
(2.10) \quad R(\xi, X)\xi &= (\alpha^2 - \beta^2 - \xi \beta)(\eta(X)\xi - X), \\
(2.11) \quad 2\alpha \beta + \xi \alpha &= 0, \\
S(X, \xi) &= [(n - 1)(\alpha^2 - \beta^2) - (\xi \beta)]\eta(X) \\
(2.12) \quad -(\phi X)\alpha - (n - 2)(\beta \beta).
\end{align}

For $\alpha, \beta = \text{constants}$ then the above equations reduce to

\begin{align}
(2.13) \quad R(\xi, X)Y &= (\alpha^2 - \beta^2)(g(X, Y)\xi - \eta(Y)X), \\
(2.14) \quad R(Y, X)\xi &= (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y), \\
(2.15) \quad S(X, Y) &= \left(\frac{r}{2} - (\alpha^2 - \beta^2)\right)g(X, Y) - \left(\frac{r}{2} - 3(\alpha^2 - \beta^2)\right)\eta(X)\eta(Y), \\
(2.16) \quad S(X, \xi) &= 2(\alpha^2 - \beta^2)\eta(X), \\
(2.17) \quad S(\phi X, \phi Y) &= \left(\frac{r}{2} - (\alpha^2 - \beta^2)\right)g(X, Y), \\
(2.18) \quad S(\phi X, Y) &= -S(X, \phi Y).
\end{align}

**Definition 2.1.** A trans-Sasakian manifold $M^n$ is said to be $\eta$-Einstein manifold if its Ricci tensor $S$ is of the form

$$
S(X, Y) = a g(X, Y) + b \eta(X)\eta(Y),
$$

where $a, b$ are smooth functions.
Let \((M^n, g)\) be a Riemannian manifold with the Levi-Civita connection \(\nabla\). A linear connection \(\tilde{\nabla}\) on \((M^n, g)\) is said to be semi-symmetric ([13], [15]) if its torsion tensor \(T\) can be written as
\[
T(X, Y) = \pi(Y)X - \pi(X)Y,
\]
where \(\pi\) is an 1--form on \(M^n\) and the associated vector field \(\rho\) defined by \(\pi(X) = g(X, \rho)\), for all vector fields \(X \in \chi(M^n)\).

A semi-symmetric connection \(\tilde{\nabla}\) is called semi-symmetric metric connection if \(\tilde{\nabla}g = 0\).

In an almost contact manifold, semi-symmetric metric connection is defined by identifying the 1-form \(\pi\) of the above equation with the contact 1-form \(\eta\), i.e., by setting [13]
\[
T(X, Y) = \eta(Y)X - \eta(X)Y,
\]
with
\[
g(X, \rho) = \eta(X), \forall X \in \chi(M^n).
\]

K. Yano has obtained the relation between semi-symmetric metric connection \(\tilde{\nabla}\) and the Levi-Civita connection \(\nabla\) of \(M^n\) in [15] and it is given by
\[
\tilde{\nabla}_XY = \nabla_XY + \eta(Y)X - g(X, Y)\xi,
\]
where \(g(X, \xi) = \eta(X)\).

Further, a relation between the curvature tensors \(R\) and \(\tilde{R}\) of type (1,3) of the connections \(\nabla\) and \(\tilde{\nabla}\), respectively is given by [15],
\[
\tilde{R}(X, Y)Z = R(X, Y)Z - K(Y, Z)X + K(X, Z)Y - g(Y, Z)F_X + g(X, Z)F_X,
\]
where \(K\) is a tensor field of type \((0,2)\) and \(F\) is a \((1,1)\) tensor field defined by
\[
K(Y, Z) = g(FY, Z) = (\nabla_Y\eta)(Z) - \eta(X)\eta(Z) + \frac{1}{2}\eta(\xi)g(Y, Z).
\]
In this paper, we have considered that $M^3$ is 3-dimensional trans-Sasakian manifold. So, using (2.9), (2.19), (2.23) it follows that

\[(2.24) \quad K(Y, Z) = -\alpha g(\phi Y, Z) - (\beta + 1)\eta(Y)\eta(Z) + (\beta + \frac{1}{2})g(Y, Z).\]

Using (2.22), from above equation we get

\[(2.25) \quad FY = -\alpha\phi Y - (\beta + 1)\eta(Y)\xi + (\beta + \frac{1}{2})Y.\]

Now, by using above two equations we get

\[(2.26) \quad \tilde{R}(X, Y)Z = R(X, Y)Z - \alpha(g(\phi X, Z)Y - g(\phi Y, Z)X) - \alpha(g(X, Z)\phi Y - g(Y, Z)\phi X)
\]

\[\quad - (\beta + 1)(\eta(X)\eta(Y) - \eta(Y)\eta(X))
\]

\[\quad - (\beta + 1)[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)]\xi
\]

\[\quad + (2\beta + 1)(g(X, Z)Y - g(Y, Z)X).\]

In the view of (2.26) we get

\[(2.27) \quad \tilde{S}(Y, Z) = S(Y, Z) + \alpha g(\phi Y, Z) + (\beta + 1)\eta(Y)\eta(Z) - (3\beta + 1)g(Y, Z),\]

where $\tilde{S}$ and $S$ are Ricci tensors of $M^3$ with respect to semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection $\nabla$, respectively.

From above, we have

\[(2.28) \quad \tilde{r} = r - 8\beta - 2,\]

where $\tilde{r}$ and $r$ are scalar curvature of $M^3$ with respect to semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection $\nabla$, respectively.

We obtain from (2.15) and (2.27) that

\[(2.29) \quad \tilde{Q}\xi = 2(\alpha^2 - \beta^2 - \beta)\xi,\]

where $\tilde{Q}$ is the Ricci operator with respect to semi-symmetric metric connection $\tilde{\nabla}$. 
3. Extended Generalized $M$-Projective $\phi$-Recurrent 3-Dimensional Trans-Sasakian Manifold with respect to Semi-Symmetric Metric Connection

**Definition 3.1.** [6] A 3-dimensional trans-Sasakian manifold is said to be a generalized $M$-projective $\phi$-recurrent manifold if the $M$-projective curvature tensor $M^*$ satisfies the relation

\[(3.1) \quad \phi^2(\nabla_W M^*)(X, Y)Z = A(W)M^*(X, Y)Z + B(W)[g(Y, Z)X - g(X, Z)Y],\]

where $A$ and $B$ are two 1-forms, $B$ is non zero and these are defined by $g(W, \rho_1) = A(W)$ and $g(W, \rho_2) = B(W), \forall W \in \chi(M)$.

And

\[(3.2) \quad M^*(X, Y)Z = R(X, Y)Z - \frac{1}{4}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY],\]

for all vector fields $X, Y, Z$ and $\rho_1$ and $\rho_2$ being the vector fields associated to the 1-form $A$ and $B$ respectively.

Next we define extended generalized $M$-projective $\phi$-recurrent manifold in the following way.

**Definition 3.2.** A 3-dimensional trans-Sasakian manifold is said to be an extended generalized $M$-projective $\phi$-recurrent manifold if the $M$-projective curvature tensor $M^*$ satisfies the relation

\[(3.3) \quad \phi^2(\nabla_W M^*)(X, Y)Z = A(W)\phi^2(M^*(X, Y)Z) + B(W)\phi^2([g(Y, Z)X - g(X, Z)Y]),\]

where $A$ and $B$ are two 1-forms, $B$ is non zero and these are defined by $g(W, \rho_1) = A(W)$ and $g(W, \rho_2) = B(W), \forall W \in \chi(M)$.

And

\[(3.4) \quad M^*(X, Y)Z = R(X, Y)Z - \frac{1}{4}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY],\]
for all vector fields \( X, Y, Z \) and \( \rho_1 \) and \( \rho_2 \) being the vector fields associated to the 1-form \( A \) and \( B \) respectively.

**Theorem 3.3.** An extended generalized \( M \)-projective \( \phi \)-recurrent three dimensional trans-Sasakian manifold with respect to semi-symmetric connection is an \( \eta \)-Einstein manifold with respect to Levi-Civita connection with \( \beta \neq -1 \) and more over, the 1-forms \( A \) and \( B \) are related as
\[
A(W)[\frac{-8\beta - 2}{4} - \frac{3}{4}(\alpha^2 - \beta^2 - \beta)] - 2B(W) = \frac{1}{4}dr(W),
\]
where \( r \) is the scalar curvature of trans-Sasakian manifold.

**Proof.** Let us assume an extended generalized \( \phi \)-recurrent trans-Sasakian manifold \((M^3, \phi, \eta, \xi, g)\) with respect to semi-symmetric connection. Then we have
\[
(3.5) \quad \phi^2((\tilde{\nabla}_W M^*)(X, Y)Z) = A(W)\phi^2(M^*(X, Y)Z) + B(W)\phi^2([g(Y, Z)X - g(X, Z)Y]).
\]

Taking inner product with \( U \) and then from the equations (2.1), (3.4), (3.5) we get
\[
-g((\tilde{\nabla}_W \tilde{R})(X, Y)Z, U) + \eta((\tilde{\nabla}_W \tilde{R})(X, Y)Z)\eta(U) + \frac{1}{4}[(\tilde{\nabla}_W \tilde{\nabla} \phi)(Y, Z)g(X, U)]
\]
\[
-(\tilde{\nabla}_W \tilde{S})(X, Z)g(Y, U) + (\tilde{\nabla}_W \tilde{S})(X, U)g(Y, Z) - (\tilde{\nabla}_W \tilde{S})(Y, U)g(X, Z)
\]
\[
-\frac{1}{4}[(\tilde{\nabla}_W \tilde{S})(Y, Z)\eta(X) - (\tilde{\nabla}_W \tilde{S})(X, Z)\eta(Y)] + (\tilde{\nabla}_W \tilde{S})(X, \xi)g(Y, Z)
\]
\[
-(\tilde{\nabla}_W \tilde{S})(Y, \xi)g(X, Z)]\eta(U) = A(W)[-g(M^*(X, Y)Z, U) + \eta(M^*(X, Y)Z)\eta(U)]
\]
\[
+B(W)[g(Y, Z)(-g(X, U) + \eta(X)\eta(U)) + g(X, Z)(g(Y, U) - \eta(Y)\eta(U))].
\]

Putting \( Z = \xi \) the equation (3.6), we have
\[
-g((\tilde{\nabla}_W \tilde{R})(X, Y)\xi, U) + \eta((\tilde{\nabla}_W \tilde{R})(X, Y)\xi)\eta(U) + \frac{1}{4}[(\tilde{\nabla}_W \tilde{\nabla} \phi)(Y, \xi)g(X, U)]
\]
\[
-(\tilde{\nabla}_W \tilde{S})(X, \xi)g(Y, U) + (\tilde{\nabla}_W \tilde{S})(X, U)g(Y, \xi) - (\tilde{\nabla}_W \tilde{S})(Y, U)g(X, \xi)
\]
\[
-\frac{1}{4}[(\tilde{\nabla}_W \tilde{S})(Y, \xi)\eta(X) - (\tilde{\nabla}_W \tilde{S})(X, \xi)\eta(Y)] + (\tilde{\nabla}_W \tilde{S})(X, \xi)g(Y, \xi)
\]
\[
-(\tilde{\nabla}_W \tilde{S})(Y, \xi)g(X, \xi)]\eta(U) = A(W)[-g(M^*(X, Y)\xi, U) + \eta(M^*(X, Y)\xi)\eta(U)]
\]
\[
+B(W)[g(Y, \xi)(-g(X, U) + \eta(X)\eta(U)) + g(X, \xi)(g(Y, U) - \eta(Y)\eta(U))].
\]
Let \( \{e_1, e_2, e_3 = \xi\} \) be an orthonormal basis for the tangent space of \( M^3 \) at a point \( p \in M^3 \). Putting \( X = U = e_i \) in (3.7) and taking summation over \( i \), we get

\[
-(\tilde{\nabla}_W \tilde{S})(Y, \xi) + \sum_{i=1}^{3} \eta((\tilde{\nabla}_W \tilde{R})(e_i, Y)\xi)\eta(e_i) + \frac{1}{4}[3(\tilde{\nabla}_W \tilde{S})(Y, \xi) - (\tilde{\nabla}_W \tilde{S})(e_i, \xi)g(Y, e_i) + (\tilde{\nabla}_W \tilde{S})(Y, \xi)g(e_i, \xi) - (\tilde{\nabla}_W \tilde{S})(e_i, \xi)g(Y, e_i)] - \frac{1}{4}[(\tilde{\nabla}_W \tilde{S})(Y, \xi)\eta(e_i) - (\tilde{\nabla}_W \tilde{S})(e_i, \xi)\eta(Y) - \eta(e_i)\eta(Y) - \eta(e_i)\eta(\xi)]
\]

(3.8)

Now,

\[
g((\tilde{\nabla}_W \tilde{R})(e_i, Y)\xi, \xi) = g((\tilde{\nabla}_W \tilde{R})(e_i, Y)\xi, \xi) + \eta(\tilde{R}(e_i, Y)\xi)g(W, \xi)
\]

(3.9)

\[-g(W, \tilde{R}(e_i, Y)\xi)g(\xi, \xi).
\]

We have

\[
g((\tilde{\nabla}_W \tilde{R})(e_i, Y)\xi, \xi) = g(\tilde{\nabla}_W \tilde{R}(e_i, Y)\xi, \xi) - g(\tilde{R}(\tilde{\nabla}_W e_i, Y)\xi, \xi)
\]

(3.10)

\[-g(\tilde{R}(e_i, \tilde{\nabla}_W Y)\xi, \xi) - g(\tilde{R}(e_i, Y)\nabla_W \xi, \xi).
\]

at \( p \in M^3 \). Since \( e_i \) is an orthonormal basis, so \( \nabla_W e_i = 0 \) at \( p \).

Also,

\[
g(\tilde{R}(e_i, Y)\xi, \xi) = -g(\tilde{R}(\xi, \xi)Y, e_i) = 0.
\]

(3.11)

Since \( \nabla_W g = 0 \), we obtain

\[
g((\tilde{\nabla}_W \tilde{R})(e_i, Y)\xi, \xi) + g(\tilde{R}(e_i, Y)\xi, \nabla_W \xi) = 0,
\]

(3.12)
which implies that

\[(3.13) \quad g((\nabla W \tilde{R})(e_i, Y)\xi, \xi) = 0.\]

Since \(\eta(\tilde{R}(e_i, Y)\xi) = 0\), we have from (3.10) that

\[(3.14) \quad g((\nabla W \tilde{R})(e_i, Y)\xi, \xi) = -g(W, \tilde{R}(e_i, Y)\xi).\]

Therefore,

\[(3.15) \quad 3 \sum_{i=1}^{3} \eta((\nabla W \tilde{R})(e_i, Y)\xi) \eta(e_i) = \alpha g(\phi Y, W) - (\alpha^2 - \beta^2 - \beta) g(\phi W, \phi Y).\]

Again, from (2.28), (3.3), (3.4), (3.15) in (3.8) we have

\[(3.16) \quad -\frac{3}{4}(\tilde{\nabla} W \tilde{S})(Y, \xi) + \frac{1}{4} dr(W)\eta(Y) + \alpha g(\phi Y, W) - (\alpha^2 - \beta^2 - \beta) g(\phi W, \phi Y)\]

\[-\alpha(\beta + 1) g(\phi Y, W)(\beta + 1)^2 \eta(W)\eta(Y) + 2(\beta + 1)(\alpha^2 - \beta^2 - \beta) \eta(Y)\eta(W)\]

\[(3.17) \quad + (3\beta + 1)(\beta + 1) g(Y, W) + 2(\alpha^2 - \beta^2 - \beta) g(\phi Y, \phi W).\]

From the equation (3.16) and (3.17) we get

\[
\begin{align*}
\alpha g(\phi Y, W) &- (\alpha^2 - \beta^2 - \beta) g(\phi W, \phi Y) + \frac{1}{4} dr(W)\eta(Y) - \frac{3}{4} [2(\alpha^2 - \beta^2 - \beta) \alpha g(\phi Y, W) \\
+ 2\beta(\alpha^2 - \beta^2 - \beta) g(\phi Y, \phi W) + \alpha S(\phi Y, W) + \alpha^2 g(\phi Y, \phi W) - (3\beta + 1) g(\phi Y, W) - \alpha(3\beta + 1) g(\phi Y, W)]
\end{align*}
\]
\[-(\beta+1)S(Y, W) - \alpha(\beta+1)g(\phi Y, W) - (\beta+1)^2 \eta(Y)\eta(W) + (3\beta+1)(\beta+1)g(Y, W) + 2(\beta + 1)(\alpha^2 - \beta^2 - \beta)\eta(Y)\eta(W) + 2(\alpha^2 - \beta^2 - \beta)g(\phi Y, \phi W)\]

\[(3.18) = (A(W)[\frac{r - 8\beta - 2}{4} - \frac{3}{2}(\alpha^2 - \beta^2)] - 2B(W))\eta(Y).\]

Replacing \(Y = \xi\) in (3.18) we obtain

\[(3.19) = \frac{1}{4} dr(W).\]

From, (3.18) we have

\[\alpha g(\phi Y, W) - (\alpha^2 - \beta^2 - \beta)g(\phi W, \phi Y) - \frac{3}{4}[2(\alpha^2 - \beta^2 - \beta)\alpha g(\phi Y, W) + 2\beta(\alpha^2 - \beta^2 - \beta)g(\phi Y, \phi W) + \alpha S(\phi Y, W) + \alpha^2 g(\phi Y, \phi W) - (3\beta + 1)g(\phi Y, W) - (\beta + 1)S(Y, W) - \alpha(\beta + 1)g(\phi Y, W) - (\beta + 1)^2 \eta(Y)\eta(W) + (3\beta + 1)(\beta + 1)g(Y, W) + 2(\beta + 1)(\alpha^2 - \beta^2 - \beta)\eta(Y)\eta(W) + 2(\alpha^2 - \beta^2 - \beta)g(\phi Y, \phi W) = 0.\]

Interchanging \(Y\) and \(W\) and then adding with the equation (3.20) the Ricci tensor is of the form

\[S(Y, W) = ag(Y, W) + b\eta(Y)\eta(W),\]

where \(a, b\) are scalar functions. Hence \(M^3\) is an \(\eta\)-Einstein manifold. \(\square\)

4. \(\phi-W_2\) flat 3-dimensional Trans-Sasakian Manifold with respect to Semi-Symmetric Metric Connection

**Definition 4.1.** Let \(M\) be 3-dimensional trans-Sasakian manifold with respect to semi-Symmetric metric connection. The \(W_2\)-curvature tensor of \(M\) is defined by

\[\bar{W}_2(X, Y)Z = \bar{R}(X, Y)Z - \frac{1}{2}(g(Y, Z)\bar{Q}X - g(X, Z)\bar{Q}Y).\]

**Definition 4.2.** A trans-Sasakian manifold \(M^3\) is said to be \(\phi - W_2\) flat with respect to semi-symmetric metric connection if

\[(4.1) \phi^2(\bar{W}_2(\phi X, \phi Y)\phi Z)) = 0.\]
Theorem 4.3. Let $M$ be a $\phi$-$\tilde{W}_2$ flat 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection. Then the manifold is an $\eta$-Einstein manifold with respect to Levi-Civita connection.

Proof. Let $M^3$ be a $\phi$-$\tilde{W}_2$ trans-Sasakian manifold with respect to semi-symmetric metric connection. It is easy to see that $\phi^2(\tilde{W}_2(\phi X, \phi Y)\phi Z)) = 0$ holds iff

$$g(\tilde{W}_2(\phi X, \phi Y)\phi Z, \phi V)) = 0, \forall X, Y, Z, V \in \chi(M^3).$$

Therefore, we get

$$g(\tilde{R}(\phi X, \phi Y)\phi Z, \phi V)) = \frac{1}{2}(\tilde{S}(\phi X, \phi Y)g(\phi Y, \phi Z) - \tilde{S}(\phi Y, \phi V)g(\phi X, \phi Z)).$$

Using the equation (4.2) in the equation (4.3), we get

$$g(\tilde{R}(\phi X, \phi Y)\phi Z, \phi V)) = \frac{1}{2}(\tilde{S}(\phi X, \phi Y)g(\phi Y, \phi Z) - \tilde{S}(\phi Y, \phi V)g(\phi X, \phi Z)).$$

Let $\{e_1, e_2, e_3 = \xi\}$ be a local orthonormal basis of vector fields in $M^3$. Then $\{\phi e_1, \phi e_2, \xi\}$ is also a local orthonormal basis of vector fields in $M^3$. Putting $X = V = e_i$ in the equation (4.4) and taking summation over $i$, $1 \leq i \leq 2$, we get,

$$\sum_{i=1}^{3} g(\tilde{R}(\phi e_i, \phi Y)\phi Z, \phi e_i)) = \frac{1}{2} \sum_{i=1}^{3} [(\tilde{S}(\phi e_i, \phi e_i)g(\phi Y, \phi Z) - \tilde{S}(\phi Y, \phi e_i)g(\phi e_i, \phi Z))].$$

Also,

$$\sum_{i=1}^{3} g(\tilde{R}(\phi e_i, Y)\phi Z, \phi e_i)) = \tilde{S}(\phi Y, \phi Z) + (\beta - \alpha^2 + \beta^2)g(\phi Y, \phi Z) + \alpha g(\phi Z, Y),$$

$$\sum_{i=1}^{3} \tilde{S}(\phi e_i, Z)g(Y, \phi e_i) = \tilde{S}(Y, Z).$$
\[(4.8) \quad \sum_{i=1}^{3} g(\phi e_i, \phi e_i) = 2,\]

\[(4.9) \quad \sum_{i=1}^{3} g(\phi e_i, Z)g(Y, \phi e_i) = g(Y, Z),\]

\[(4.10) \quad \sum_{i=1}^{3} g(\phi e_i, e_i) = 0.\]

Therefore, using the equations (4.6), (4.7), (4.8), (4.9) and (4.10) we have

\[(4.11) \quad \tilde{S}(\phi Y, \phi Z) + (\beta - \alpha^2 + \beta^2)g(\phi Y, \phi Z) + \alpha g(\phi Z, Y) = \frac{1}{2} [g(\phi Y, \phi Z)\tilde{r} - \tilde{S}(\phi Y, \phi Z)].\]

Then, we have

\[(4.12) \quad 3\tilde{S}(\phi Y, \phi Z) = -2\alpha g(\phi Z, Y) + g(\phi Y, \phi Z)(\tilde{r} - 2(\beta - \alpha^2 + \beta^2)).\]

Hence from the above equation (4.12) we get

\[(4.13) \quad 3S(\phi Y, \phi Z) + 3\alpha g(\phi^2 Y, \phi Z) - 3(3\beta + 1)g(\phi Y, \phi Z) = -2\alpha g(\phi Z, Y)
+ g(\phi Y, \phi Z)(r - 8\beta - 2 - 2(\beta - \alpha^2 + \beta^2)),\]

Therefore, we get

\[(4.14) \quad 3S(Y, Z) = (r - 9\beta + 1 + 2\alpha^2 - 2\beta^2)g(Y, Z) + (4\alpha^2 - 4\beta^2 + 9\beta - 1 - r)\eta(Y)\eta(Z) + \alpha g(Y, \phi Z).\]

Interchanging \(Y\) with \(Z\) in (4.14) we get

\[(4.15) \quad 3S(Z, Y) = (r - 9\beta + 1 + 2\alpha^2 - 2\beta^2)g(Z, Y) + (4\alpha^2 - 4\beta^2 + 9\beta - 1 - r)\eta(Z)\eta(Y) + \alpha g(Z, \phi Y).\]

Then adding above two equations and using skew-symmetric property
of $\phi$ we have

$$(4.16) \quad S(Y, Z) = \frac{1}{3}(r-9\beta+1+2\alpha^2-2\beta^2)g(Y, Z)+\frac{1}{3}(4\alpha^2-4\beta^2+9\beta-1-r)\eta(Y)\eta(Z).$$

This proves that $M^3$ is an $\eta$-Einstein manifold.

**Corollary 4.4.** Let $M$ be a $\phi$-$\tilde{W}_2$ flat 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection. Then the manifold is a $\eta$-Einstein manifold with respect to semi-symmetric metric connection if $\alpha = 0$ i.e. if $M$ is a $\beta$-Kenmotsu manifold.

5. Conharmonically Flat 3-dimensional Trans-Sasakian Manifold with respect to Semi-Symmetric Metric Connection

A Conharmonic curvature tensor has been studied by Ozgur [11], Siddiqui and Ahsan [12] and many other authors. In almost contact manifold $M$ of dimension $n \geq 3$, the conharmonic curvature tensor $\tilde{K}$ with respect to semi-symmetric connection $\tilde{\nabla}$ is given by

$$(5.1) \quad \tilde{K}(X, Y)Z = \tilde{R}(X, Y)Z - \frac{1}{n-2}[\tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y + \tilde{Q}X - \tilde{Q}Y],$$

for $X, Y, Z \in \chi(M)$, where $\tilde{R}, \tilde{S}, \tilde{Q}$ are the Riemannian curvature tensor, Ricci tensor and the Ricci operator with respect to semi-symmetric connection $\tilde{\nabla}$ respectively.

A conharmonic curvature tensor $\tilde{K}$ with respect to semi-symmetric connection $\tilde{\nabla}$ is said to flat if it vanishes identically with respect to semi-symmetric connection $\tilde{\nabla}$.

Now, we prove the following theorem.

**Theorem 5.1.** Let $M$ be a conharmonically flat 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection. Then the manifold is a $\eta$-Einstein manifold with respect to Levi-Civita connection.
Proof. Assume that \( M \) is a conharmonically flat 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection. Then from the equation (5.1) we get

\[
\tilde{R}(X,Y)Z = [\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y + g(Y,Z)\tilde{Q}X - g(X,Z)\tilde{Q}Y].
\]

Then we have

\[
g(\tilde{R}(X,Y)Z,U) = [\tilde{S}(Y,Z)g(X,U) - \tilde{S}(X,Z)g(Y,U) + g(Y,Z)\tilde{S}(X,U) - g(X,Z)\tilde{S}(Y,U)].
\]

Putting \( X = U = \xi \) in the above the equation (5.3) we get

\[
\tilde{S}(Y,Z) = [-\alpha^2 - \beta^2 - \beta]g(Y,Z) + 3(\alpha^2 - \beta^2 - \beta)\eta(Y)\eta(Z) + \alpha g(\phi Y, Z).
\]

Hence from the above the equation and using the equation (2.27) we get

\[
S(Y,Z) = (4\beta + 1 - \alpha^2 + \beta^2)g(Y,Z) + (3\alpha^2 - 3\beta^2 - 4\beta - 1)\eta(Y)\eta(Z) - 2\alpha g(\phi Y, Z).
\]

Interchanging \( Y \) with \( Z \) in (5.5) we get

\[
S(Z,Y) = (4\beta + 1 - \alpha^2 + \beta^2)g(Z,Y) + (3\alpha^2 - 3\beta^2 - 4\beta - 1)\eta(Z)\eta(Y) - 2\alpha g(\phi Z, Y).
\]

Adding the above two equations we obtain

\[
S(Y,Z) = (4\beta + 1 - \alpha^2 + \beta^2)g(Y,Z) + (3\alpha^2 - 3\beta^2 - 4\beta - 1)\eta(Y)\eta(Z).
\]

Hence, the manifold is an \( \eta \)-Einstein manifold with respect to Levi-Civita connection. Therefore, the theorem is proved.
Corollary 5.2. Let $M$ be a conharmonically flat 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection. Then the manifold is an $\eta$-Einstein manifold with respect to semi-symmetric metric connection if $\alpha = 0$ i.e. if $M$ is a $\beta$-Kenmotsu manifold.

6. $\phi$-Conharmonically Flat 3-dimensional Trans-Sasakian Manifold with respect to Semi-Symmetric Metric Connection

Definition 6.1. A trans-Sasakian manifold $M^3$ is said to be $\phi$-conharmonically flat with respect to semi-symmetric connection if

\begin{equation}
\phi^2(\tilde{K}(\phi X, \phi Y)\phi Z)) = 0,
\end{equation}

where $\tilde{K}$ is the conharmonic curvature tensor with respect to semi-symmetric metric connection.

Theorem 6.2. Let $M$ be a $\phi$-conharmonically flat 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection. Then the manifold is a $\eta$-Einstein manifold with respect to Levi-Civita connection.

Proof. Let $M^3$ be a $\phi$-$\tilde{K}$ be a 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection. It is easy to see that $\phi^2(\tilde{K}(\phi X, \phi Y)\phi Z)) = 0$ holds iff

\begin{equation}
g(\tilde{K}(\phi X, \phi Y)\phi Z, \phi V) = 0, \forall X, Y, Z, V \in \chi(M^3).
\end{equation}

Now, from the definition of conharmonic curvature tensor on 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connection, we get

\begin{equation}
g(\tilde{K}(\phi X, \phi Y)\phi Z, \phi V)) = g(\tilde{R}(\phi X, \phi Y)\phi Z, \phi V)) - [\tilde{S}(\phi Y, \phi Z)g(\phi X, \phi V)
- \tilde{S}(\phi X, \phi Z)g(\phi Y, \phi V)]
\end{equation}

Using the equation (6.2) in the equation (6.3), we get

\begin{equation}
g(\tilde{R}(\phi X, \phi Y)\phi Z, \phi V)) = [\tilde{S}(\phi Y, \phi Z)g(\phi X, \phi V) - \tilde{S}(\phi X, \phi Z)g(\phi Y, \phi V)
+ g(\phi Y, \phi Z)\tilde{S}(\phi X, \phi V) - g(\phi X, \phi Z)\tilde{S}(\phi Y, \phi V)].
\end{equation}
Let \( \{e_1,e_2,e_3 = \xi\} \) be a local orthonormal basis of vector fields in \( M^3 \). Then \( \{\phi e_1, \phi e_2, \xi\} \) is also a local orthonormal basis of vector fields in \( M^3 \). Putting \( X = V = e_i \) in the equation (6.4) and taking summation over \( i, \ 1 \leq i \leq 3 \), we get

\[
g(\tilde{R}(\phi e_i, \phi Y)\phi Z, \phi g_i)) = [\tilde{S}(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - \tilde{S}(\phi e_i, \phi Z)g(\phi Y, \phi e_i) - g(\phi e_i, \phi Z)\tilde{S}(\phi Y, \phi e_i) - g(\phi Y, \phi Z)\tilde{S}(\phi e_i, \phi e_i)]
\]

Therefore, using the equations (4.6)-(4.10) we have

\[
\tilde{S}(\phi Y, \phi Z) + \alpha g(Y, \phi Z) = [\tilde{r} + (\beta - \alpha^2 + \beta^2)]g(\phi Y, \phi Z).
\]

Hence from the equations (2.13), (2.17), (2.27) we get

\[
S(Y, Z) = [r - 6\beta - 1 + \alpha^2 - \beta^2]g(Y, Z) + (\alpha^2 - \beta^2 + 6\beta - r + 1)\eta(Y)\eta(Z).
\]

This proves that \( M^3 \) is an \( \eta \)-Einstein manifold with respect to Levi-Civita connection.

\[\square\]

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**References**


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