GENERIC LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE TRANS-SASAKIAN MANIFOLD WITH A QUARTER-SYMMETRIC METRIC CONNECTION

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Abstract. The object of study in this paper is generic lightlike submanifolds of an indefinite trans-Sasakian manifold with a quarter-symmetric metric connection. We study the geometry of two types of generic lightlike submanifolds, which are called recurrent and Lie recurrent generic lightlike submanifolds, of an indefinite trans-Sasakian manifold with a quarter-symmetric metric connection.

1. Introduction

Yano-Imai [17] introduced the notion of quarter-symmetric metric connection on a Riemannian manifold. Recently, Jin [7, 10] studied the geometry of lightlike hypersurfaces of an indefinite trans-Sasakian manifold with a quarter-symmetric metric connection. We quote Jin’s definition in itself as follow:

A linear connection $\overline{\nabla}$ on a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is said to be a quarter-symmetric metric connection if it is metric, i.e., $\overline{\nabla}\overline{g} = 0$ and its torsion tensor $\overline{T}$, defined by $\overline{T}(\overline{X}, \overline{Y}) = \overline{\nabla}_X \overline{Y} - \overline{\nabla}_Y \overline{X} - [\overline{X}, \overline{Y}]$, satisfies

\begin{equation}
\overline{T}(\overline{X}, \overline{Y}) = \theta(\overline{Y})J\overline{X} - \theta(\overline{X})J\overline{Y},
\end{equation}

where $J$ is a $(1,1)$-type tensor field on $\overline{M}$ and $\theta$ is a 1-form associated with a smooth unit vector field $\zeta$ on $\overline{M}$ by $\theta(X) = \overline{g}(X, \zeta)$. Throughout this paper, we denote by $\overline{X}, \overline{Y}$ and $\overline{Z}$ the smooth vector fields on $\overline{M}$.

A lightlike submanifold $M$ of an indefinite almost contact manifold $\overline{M}$ is called generic if there exists a screen distribution $S(TM)$ of $M$ such that

\begin{equation}
J(S(TM)^\perp) \subset S(TM),
\end{equation}

where $S(TM)^\perp$ is the orthogonal complement of $S(TM)$ in the tangent bundle $TM$ of $\overline{M}$, i.e., $TM = S(TM) \oplus_{\text{orth}} S(TM)^\perp$. The notion of generic lightlike submanifolds was introduced by Jin-Lee [12] at 2011 and then, studied by Duggal-Jin [5], Jin [6, 8] and Jin-Lee [14] and several authors. The geometry of...
generic lightlike submanifolds is an extension of that of lightlike hypersurface and half lightlike submanifold of codimension 2, that is, the last two types of lightlike submanifolds are examples of the generic lightlike submanifold. Much of the theory of generic lightlike submanifolds will be immediately generalized in a formal way to general lightlike submanifolds.

The notion of trans-Sasakian manifold, of type \((\alpha, \beta)\), was introduced by Oubina [16]. If a trans-Sasakian manifold \(\tilde{M}\) is semi-Riemannian, then \(\tilde{M}\) is called an indefinite trans-Sasakian manifold. Sasakian, Kenmotsu and cosymplectic manifolds are important kinds of trans-Sasakian manifold such that \(\alpha = 1, \beta = 0; \alpha = 0, \beta = 1; \alpha = \beta = 0\), respectively.

The object of study of this paper is generic lightlike submanifolds of an indefinite trans-Sasakian manifold \(\tilde{M}\) with respect to the semi-Riemannian metric \(\tilde{g}\). It is known [9] that a linear connection \(\tilde{\nabla}\) on \(\tilde{M}\) is a quarter-symmetric metric connection if and only if \(\tilde{\nabla}\) satisfies

\[
\nabla_X Y = \tilde{\nabla}_X Y - \theta(X)JY. \tag{1.3}
\]

2. Preliminaries

An odd-dimensional semi-Riemannian manifold \((\tilde{M}, \tilde{g})\) is called an indefinite trans-Sasakian manifold if there exist (1) a structure set \(\{J, \zeta, \theta, \tilde{g}\}\), where \(J\) is a \((1,1)\)-type tensor field, \(\zeta\) is a vector field and \(\theta\) is a 1-form such that

\[
\begin{align*}
J^2 \tilde{X} &= -\tilde{X} + \theta(\tilde{X})\zeta, \quad \theta(\zeta) = 1, \quad \theta(\tilde{X}) = \epsilon \tilde{g}(\tilde{X}, \zeta), \\
\theta \circ J &= 0, \quad \tilde{g}(J\tilde{X}, J\tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{Y}) - \epsilon \theta(\tilde{X})\theta(\tilde{Y}),
\end{align*}
\]

(2.1)

and (2) two smooth functions \(\alpha\) and \(\beta\), and a Levi-Civita connection \(\tilde{\nabla}\) such that

\[
\tilde{\nabla}_X J\tilde{Y} = \alpha\{\tilde{g}(\tilde{X}, \tilde{Y})\zeta - \epsilon \theta(\tilde{Y})\tilde{X}\} + \beta\{\tilde{g}(J\tilde{X}, \tilde{Y})\zeta - \epsilon \theta(\tilde{Y})J\tilde{X}\},
\]

where \(\epsilon\) denotes \(\epsilon = 1\) or \(-1\) according as \(\zeta\) is spacelike or timelike respectively. \(\{J, \zeta, \theta, \tilde{g}\}\) is called an indefinite trans-Sasakian structure of type \((\alpha, \beta)\).

In the entire discussion of this article, we shall assume that the vector field \(\zeta\) is a spacelike one, i.e., \(\epsilon = 1\), without loss of generality.

By directed calculation from (1.3), we see that \((\tilde{\nabla}_X J)\tilde{Y} = (\tilde{\nabla}_X J)\tilde{Y}\). Thus, replacing the Levi-Civita connection \(\tilde{\nabla}\) by the quarter-symmetric metric connection \(\nabla\) defined by (1.3), the last equation is reformed to

\[
(\nabla_X J)\tilde{Y} = \alpha\{\tilde{g}(\tilde{X}, \tilde{Y})\zeta - \theta(\tilde{Y})\tilde{X}\} + \beta\{\tilde{g}(J\tilde{X}, \tilde{Y})\zeta - \theta(\tilde{Y})J\tilde{X}\}. \tag{2.2}
\]

Replacing \(Y\) by \(\zeta\) to (2.2) and using \(J\zeta = 0\) and \(\theta(\nabla_X \zeta) = 0\), we obtain

\[
\nabla_X \zeta = -\alpha J\tilde{X} + \beta(\tilde{X} - \theta(\tilde{X})\zeta). \tag{2.3}
\]
Let \((M, g)\) be an \(m\)-dimensional lightlike submanifold of an indefinite trans-Sasakian manifold \((\bar{M}, \bar{g})\) of dimension \((m + n)\). Then the radical distribution \(\text{Rad}(TM) = TM \cap TM^\perp\) of \(M\) is a subbundle of the tangent bundle \(TM\) and the normal bundle \(TM^\perp\), of rank \(r (1 \leq r \leq \min\{m, n\})\). In general, there exist two complementary non-degenerate distributions \(S(TM)\) and \(S(TM^\perp)\) of \(\text{Rad}(TM)\) in \(TM\) and \(TM^\perp\) respectively, which are called the screen distribution and the co-screen distribution of \(M\), such that

\[
TM = \text{Rad}(TM) \oplus_{\text{orth}} S(TM), \quad TM^\perp = \text{Rad}(TM) \oplus_{\text{orth}} S(TM^\perp),
\]

where \(\oplus_{\text{orth}}\) denotes the orthogonal direct sum. Denote by \(F(M)\) the algebra of smooth functions on \(M\) and by \(\Gamma(E)\) the \(F(M)\) module of smooth sections of a vector bundle \(E\) over \(M\). Also denote by (2.1) the \(i\)-th equation of (2.1).

We use the same notations for any others. Let \(X, Y, Z\) and \(W\) be the vector fields on \(M\), unless otherwise specified. We use the following range of indices:

\[
i, j, k, \ldots, \in \{1, \ldots, r\}, \quad a, b, c, \ldots, \in \{r + 1, \ldots, n\}.
\]

Let \(\text{tr}(TM)\) and \(\text{ltr}(TM)\) be complementary vector bundles to \(TM\) in \(TM|_M\) and \(TM^\perp\) in \(S(TM)^\perp\) respectively and let \(\{N_1, \ldots, N_r\}\) be a lightlike basis of \(\text{ltr}(TM)|_U\), where \(U\) is a coordinate neighborhood of \(M\), such that

\[
\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0,
\]

where \(\{\xi_1, \ldots, \xi_r\}\) is a lightlike basis of \(\text{Rad}(TM)|_U\). Then we have

\[
TM = TM \oplus \text{tr}(TM) = \{\text{Rad}(TM) \oplus \text{tr}(TM)\} \oplus_{\text{orth}} S(TM) = \{\text{Rad}(TM) \oplus \text{ltr}(TM)\} \oplus_{\text{orth}} S(TM) \oplus_{\text{orth}} S(TM^\perp).
\]

We say that a lightlike submanifold \((M, g, S(TM), S(TM^\perp))\) of \(\bar{M}\) is

1. \(r\)-lightlike submanifold if \(1 \leq r < \min\{m, n\}\);
2. co-isotropic submanifold if \(1 \leq r = n < m\);
3. isotropic submanifold if \(1 \leq r = m < n\);
4. totally lightlike submanifold if \(1 \leq r = m = n\).

The above three classes (2)–(4) are particular cases of the class (1) as follows:

\[
S(TM^\perp) = \{0\}, \quad S(TM) = \{0\}, \quad S(TM) = S(TM^\perp) = \{0\}
\]

respectively. The geometry of \(r\)-lightlike submanifolds is more general than that of the other three types. For this reason, we consider only \(r\)-lightlike submanifolds \(M\), with following local quasi-orthonormal field of frames of \(M\):

\[
\{\xi_1, \ldots, \xi_r, N_1, \ldots, N_r, F_{r+1}, \ldots, F_m, E_{r+1}, \ldots, E_n\},
\]

where \(\{F_{r+1}, \ldots, F_m\}\) and \(\{E_{r+1}, \ldots, E_n\}\) are orthonormal bases of \(S(TM)\) and \(S(TM^\perp)\), respectively. Denote \(\epsilon_a = \bar{g}(E_a, E_a)\). Then \(\epsilon_a \delta_{ab} = \bar{g}(E_a, E_b)\).

Let \(P\) be the projection morphism of \(TM\) on \(S(TM)\). Then the local Gauss-Weingarten formulas of \(M\) and \(S(TM)\) are given respectively by

\[
\nabla_X Y = \nabla_X Y + \sum_{i=1}^r b_i^r(X, Y)N_i + \sum_{a=r+1}^n h_a^r(X, Y)E_a,
\]

(2.4)
Applying \( J \) linear operators on \( T_M \)

\[
\nabla_X N_i = -A_{N_i} X + \sum_{j=1}^{r} \tau_{ij}(X) N_j + \sum_{a=r+1}^{n} \rho_a(X) E_a,
\]

\[
\nabla_X E_a = -A_{E_a} X + \sum_{i=1}^{r} \phi_{ai}(X) N_i + \sum_{b=r+1}^{n} \sigma_{ab}(X) E_b;
\]

\[
\nabla_X PY = \nabla_X^* PY + \sum_{i=1}^{r} h^*_i(X, PY) \xi_i,
\]

where \( \nabla \) and \( \nabla^* \) are induced linear connections on \( M \) and \( S(TM) \) respectively, \( h_i^* \) and \( h^*_i \) are called the local second fundamental forms on \( M \), \( h_i^* \) are called the local screen second fundamental forms on \( S(TM) \). \( A_{N_i}, A_{E_a} \) and \( A^*_i \) are linear operators on \( M \), and \( \tau_{ij}, \rho_a, \phi_{ai}, \sigma_{ab} \) are 1-forms on \( M \).

3. Quarter-symmetric metric connection

Now we assume that \( \zeta \) is tangent to \( M \). Călin \[2\] proved that if \( \zeta \) is tangent to \( M \), then it belongs to \( S(TM) \) which we assume. For a generic \( M \), from (1.2) we show that \( J(\text{Rad}(TM)), J(\text{ltr}(TM)) \) and \( J(S(TM^\perp)) \) are subbundles of \( S(TM) \). Thus there exist two non-degenerate almost complex distributions \( H_o \) and \( H \) with respect to \( J \), i.e., \( J(H_o) = H_o \) and \( J(H) = H \), such that

\[
S(TM) = \{ J(\text{Rad}(TM)) \oplus J(\text{ltr}(TM)) \} \oplus_{\text{orth}} J(S(TM^\perp)) \oplus_{\text{orth}} H_o,
\]

\[
H = \text{Rad}(TM) \oplus_{\text{orth}} J(\text{Rad}(TM)) \oplus_{\text{orth}} H_o.
\]

In this case, the tangent bundle \( TM \) of \( M \) is decomposed as follow:

\[
TM = H \oplus J(\text{ltr}(TM)) \oplus_{\text{orth}} J(S(TM^\perp)).
\]

Consider local null vector fields \( U_i \) and \( V_i \) for each \( i \), local non-null unit vector fields \( W_a \) for each \( a \), and their 1-forms \( u_i, v_i \) and \( w_a \) defined by

\[
U_i = -JN_i, \quad V_i = -J\xi_i, \quad W_a = -JE_a;
\]

\[
u_i(X) = g(X, V_i), \quad v_i(X) = g(X, U_i), \quad w_a(X) = \epsilon_a g(X, W_a).
\]

Denote by \( S \) the projection morphism of \( TM \) on \( H \) and by \( F \) the tensor field of type \((1,1)\) globally defined on \( M \) by \( F = J \circ S \). Then \( JX \) is expressed as

\[
JX = FX + \sum_{i=1}^{r} u_i(X) N_i + \sum_{a=r+1}^{n} w_a(X) E_a.
\]

Applying \( J \) to (3.4) and using (2.1)\_1 and (3.2), we have

\[
F^2X = -X + \theta(X) \zeta + \sum_{i=1}^{r} u_i(X) U_i + \sum_{a=r+1}^{n} w_a(X) W_a.
\]

In the following, we say that \( F \) is the structure tensor field of \( M \).
Substituting (2.4) and (3.4) into (1.1) and then, comparing the tangent, lightlike transversal and co-screen components of the left-right terms, we get

\[ T(X, Y) = \theta(Y)FX - \theta(X)FY, \]
\[ h^l_t(X, Y) = h^l_t(Y, X) = \theta(Y)u_i(X) - \theta(X)u_t(Y), \]
\[ h^a_t(X, Y) = h^a_t(Y, X) = \theta(Y)w_a(X) - \theta(X)w_a(Y), \]

where \( T \) is the torsion tensor with respect to the connection \( \nabla \). Note that, from (3.7) and (3.8), we see that \( h^l_t \) and \( h^a_t \) are not symmetric.

From the facts that \( h^l_t(X, Y) = \hat{g}(\nabla_X Y, \xi) \) and \( \epsilon_a h^a_t(X, Y) = \hat{g}(\nabla_X Y, E_a) \), we know that \( h^l_t \) and \( h^a_t \) are independent of the choice of \( S(TM) \). The local second fundamental forms are related to their shape operators by

\[ h^l_t(X, Y) = g(A^l_t, X, Y) - \sum_{k=1}^r h^e_k(X, \xi_k)\eta_k(Y), \]
\[ \epsilon_a h^a_t(X, Y) = g(A^a_t, X, Y) - \sum_{k=1}^r \phi_{ak}(X)\eta_k(Y), \]
\[ h^e_t(X, PY) = g(A^e_t, X, PY), \]

where \( \eta_k \)'s are 1-forms such that \( \eta_k(X) = \hat{g}(X, N_k) \). Applying \( \nabla_X \) to \( g(\xi, \xi) = 0, \hat{g}(\xi, E_a) = 0, \hat{g}(N_i, E_j) = 0, \hat{g}(E_a, E_b) = 0 \), we obtain

\[ h^l_t(X, \xi_j) + h^l_t(X, \xi_i) = 0, \quad h^a_t(X, \xi_i) = -\epsilon_a \phi_{ai}(X), \]
\[ \eta_j(A^l_t, X) + \eta_i(A^l_t, X) = 0, \quad g(A^l_t, X, N_i) = \epsilon_a \rho_{ai}(X), \]
\[ \epsilon_b \sigma_{ab} + \epsilon_a \sigma_{ba} = 0 \quad \text{and} \quad h^l_t(X, \xi_i) = 0, \quad h^l_t(\xi_j, \xi_k) = 0. \]

By directed calculations from (2.3), (2.4), (2.5), (3.4) and (3.11), we have

\[ \nabla_X \zeta = -\alpha FX + \beta(X - \theta(X))\zeta, \]
\[ h^l_t(X, \zeta) = -\alpha w_t(X), \quad h^a_t(X, \zeta) = -\alpha w_a(X), \]
\[ h^e_t(X, \zeta) = -\alpha w_t(X) + \beta \eta_t(X). \]

Applying \( \nabla_X \) to (3.2), (3.3) and (3.4) by turns and using (2.2), (2.4) \sim (2.8), (2.2) \sim (3.4) and (3.9) \sim (3.11), we have

\[ h^l_t(X, U_i) = h^l_t(X, V_j), \quad \epsilon_a h^a_t(X, W_a) = h^a_t(X, U_i), \]
\[ h^e_t(X, V_i) = h^e_t(X, V_j), \quad \epsilon_a h^a_t(X, W_a) = h^a_t(X, V_i), \]
\[ \epsilon_b h^b_t(X, W_b) = \epsilon_a h^a_t(X, W_b), \]
\[ \nabla_X U_i = F(A^l_t, X) + \sum_{j=1}^r \tau_{ij}(X)U_j + \sum_{a=r+1}^n \rho_{ia}(X)W_a \]
\[ - (\alpha \eta_t(X) + \beta \nu_t(X))\zeta, \]
\[ \nabla_X V_i = F(A^e_t, X) - \sum_{j=1}^r \tau_{ji}(X)V_j + \sum_{j=1}^r h^l_t(X, \xi_i)U_j \]
Remark 4.1. From (2.4) and (3.12)2, the item (1) is equivalent to

\begin{equation}
(4.1) \quad h^i_j(X, \xi) = 0, \quad h^*_{ab}(X, \xi) = \phi_{ai}(X) = 0.
\end{equation}

By using (3.12)4, the item (2) is equivalent to

\begin{equation}
(4.2) \quad \eta_j(A_{N_i} X) = 0, \quad \rho_{ab}(X) = \eta_i(A_{E_a} X) = 0.
\end{equation}

Denote by \(\lambda_{ij}\), \(\mu_{ab}\), \(\nu_{ab}\), \(\kappa_{ab}\) and \(\chi_{ij}\) the 1-forms on \(M\) such that

\begin{equation}
(4.3) \quad \lambda_{ij}(X) = h^i_j(X, U_j) = h^*_{ij}(X, V_i), \quad \kappa_{ab}(X) = \epsilon_a h^*_{ab}(X, W_b),
\end{equation}

\begin{equation}
\mu_{ab}(X) = h^i_j(X, W_a) = \epsilon_a h^*_{ab}(X, V_i), \quad \chi_{ij}(X) = h^i_j(X, V_j),
\end{equation}

\begin{equation}
\nu_{ab}(X) = h^*_{ij}(X, W_a) = \epsilon_a h^*_{ab}(X, U_i).
\end{equation}
Definition. The structure tensor field $F$ of $M$ is said to be recurrent [11] if there exists a 1-form $\varpi$ on $M$ such that

$$ (\nabla_X F)Y = \varpi(X)FY. $$

A lightlike submanifold $M$ of an indefinite trans-Sasakian manifold $\bar{M}$ is called recurrent if it admits a recurrent structure tensor field $F$.

Theorem 4.2. Let $M$ be a recurrent generic lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection. Then the following statements are satisfied:

1. $F$ is parallel with respect to the induced connection $\nabla$ on $M$,
2. $M$ is an indefinite cosymplectic manifold, i.e., $\alpha = \beta = 0$,
3. $M$ is statical,
4. $J(ltr(TM))$, $J(S(TM^{-1}))$ and $H$ are parallel distributions on $M$,
5. $M$ is locally a product manifold $M_r \times M_{n-r} \times M^s$, where $M_r, M_{n-r}$ and $M^s$ are leaves of $J(ltr(TM))$, $J(S(TM^{-1}))$ and $H$, respectively.

Proof. (1) From the above definition and (3.20), we obtain

$$ (4.4) \quad \varpi(X)FY = \sum_{i=1}^{r} u_i(Y)A_{\alpha_i}X + \sum_{a=r+1}^{n} w_a(Y)A_{\mu_a}X $$

$$ - \sum_{i=1}^{r} h^i(X,Y)U_i - \sum_{a=r+1}^{n} h^a_\alpha(X,Y)W_a $$

$$ + \alpha\{g(X,Y)\zeta - \theta(Y)X\} + \beta\{\bar{g}(JX,Y)\zeta - \theta(Y)FX\}. $$

Replacing $Y$ by $\xi_j$ and using the fact that $F\xi_j = -V_j$, we get

$$ (4.5) \quad \varpi(X)V_j = \sum_{k=1}^{r} h^k(X,\xi_j)U_k + \sum_{b=r+1}^{n} h^b_\alpha(X,\xi_j)W_b - \beta u_j(X)\zeta. $$

Taking the scalar product with $U_j, \zeta, V_i$ and $W_a$ by turns, we obtain

$$ \varpi = 0, \quad \beta = 0, \quad h^i(X,\xi_j) = 0, \quad h^a_\alpha(X,\xi_j) = \phi_{aj}(X) = 0, $$

respectively. As $\varpi = 0$, $F$ is parallel with respect to the connection $\nabla$.

(2) Taking the scalar product with $U_j$ to (4.4) with $\varpi = 0$, we get

$$ (4.6) \quad \sum_{i=1}^{r} u_i(Y)g(A_{\alpha_i}X, U_j) + \sum_{a=r+1}^{n} w_a(Y)g(A_{\mu_a}X, U_j) - \alpha \theta(Y)v_j(X) = 0. $$

Replacing $Y$ by $\zeta$ to this equation, we have $\alpha v_j(X) = 0$. It follows that $\alpha = 0$. As $\alpha = \beta = 0$, $\bar{M}$ is an indefinite cosymplectic manifold.

(3) As $h^i(X,\xi_j) = 0$ and $h_a^\alpha(X,\xi_j) = 0$, $M$ is irrotational by (4.1). Also, $M$ is solenoidal. In fact, taking the scalar product with $N_j$ to (4.4), we have

$$ \sum_{i=1}^{r} u_i(Y)\bar{g}(A_{\alpha_i}X, N_j) + \sum_{a=r+1}^{n} w_a(Y)\bar{g}(A_{\mu_a}X, N_j) = 0. $$
Taking \( Y = U_j \) and \( Y = W_a \) by turns, we get (4.2). Thus \( M \) is statical.

(4) Taking \( Y = U_k \) and \( Y = W_b \) to (4.6) by turns, we obtain

\[
(4.7) \quad h^i_\nu(X, U_j) = g(A_{\nu i} X, U_j) = 0, \quad \nu a_i(X) = g(A_{\nu a} X, U_i) = 0.
\]

Taking the scalar product with \( V_j \) and \( W_b \) to (4.4) by turns, we have

\[
(4.8) \quad h^i_\nu(X, Y) = \sum_{j=1}^r \lambda_{ij}(X) u_j(Y) + \sum_{a=r+1}^n \mu_{\nu a}(X) w_a(Y),
\]

\[
\epsilon_a h^a_{\nu}(X, Y) = \sum_{b=r+1}^n \kappa_{ba}(X) w_b(Y),
\]

due to (3.10), (3.11) and (4.3). Replacing \( Y \) by \( V_j \) to (4.8)\(_{1,2} \), we have

\[
(4.9) \quad \lambda_{ij}(X) = h^i_\nu(X, V_j) = 0, \quad \mu_{\nu a}(X) = h^a_{\nu}(X, V_i) = 0.
\]

Taking \( Y = U_j \) and \( Y = W_b \) to (4.4) and using (4.3), (4.7)\(_2 \) and (4.9)\(_2 \), we get

\[
(4.10) \quad A_{\nu i} X = \sum_{j=1}^r \lambda_{ij}(X) U_j, \quad A_{\nu a} X = \sum_{b=r+1}^n \epsilon_b \kappa_{ba}(X) W_b.
\]

Using (3.9), (4.1), (4.9)\(_2 \) and the non-degenerateness of \( S(TM) \), (4.8)\(_1 \) reduces

\[
(4.11) \quad A^\nu_i X = \sum_{j=1}^r \lambda_{ij}(X) V_j.
\]

Applying \( F \) to (4.10)\(_{1,2} \), we have \( F(A_{\nu i} X) = 0 \) and \( F(A_{\nu a} X) = 0 \). Substituting these results into (3.17) and (3.19), we obtain

\[
(4.12) \quad \nabla_X U_i = \sum_{j=1}^r \tau_{ij}(X) U_j, \quad \nabla_X W_a = \sum_{b=r+1}^n \sigma_{ab}(X) W_b.
\]

It follows that \( J(ltr(TM)) \) and \( J(S(TM^\perp)) \) are parallel distributions on \( M \) with respect to the induced connection \( \nabla \) on \( M \), that is,

\[
\nabla_X U_i \in \Gamma(J(ltr(TM))), \quad \nabla_X W_a \in \Gamma(J(S(TM^\perp))).
\]

Applying \( F \) to (4.11), we get \( F(A^\nu_i X) = \sum_{j=1}^r \lambda_{ij}(X) \xi_j \), thus we have

\[
(4.13) \quad \nabla_X V_i = \sum_{j=1}^r \{ \lambda_{ij}(X) \xi_j - \tau_{ij}(X) V_j \}.
\]

Taking \( Y \in \Gamma(H) \) to (4.4) and then, taking the scalar product with \( U_j \) and \( W_b \) to the resulting equation by turns, we obtain

\[
(4.14) \quad h^i_\nu(X, Y) = 0, \quad h^a_{\nu}(X, Y) = 0, \quad \forall X \in \Gamma(TM), \quad \forall Y \in \Gamma(H).
\]

By directed calculations from (4.9), (4.12)\(_2 \), (4.13) and (4.14), we obtain \( g(\nabla_X Y, V_i) = 0 \) and \( g(\nabla_X Y, W_a) = 0 \) for all \( X \in \Gamma(TM) \) and \( Y \in \Gamma(H) \). Thus

\[
\nabla_X Y \in \Gamma(H), \quad \forall X \in \Gamma(TM), \quad \forall Y \in \Gamma(H).
\]
Thus $H$ is also a parallel distribution on $M$ with respect to $\nabla$.

(5) As $J(ltr(TM))$, $J(S(TM^\perp))$ and $H$ are parallel distributions and satisfied the decomposition form (3.1), by the decomposition theorem of de Rham [3], $M$ is locally a product manifold $M_r \times M_{n-r} \times M^\sharp$, where $M_r$, $M_{n-r}$ and $M^\sharp$ are leaves of $J(ltr(TM))$, $J(S(TM^\perp))$ and $H$, respectively. □

**Definition.** The structure tensor field $F$ of $M$ is said to be **Lie recurrent** [11] if there exists a 1-form $\vartheta$ on $M$ such that

$$ (\mathcal{L}_X F) Y = \vartheta(X) FY, $$

where $\mathcal{L}_X$ denotes the Lie derivative on $M$ with respect to $X$. The structure tensor field $F$ is called **Lie parallel** if $\mathcal{L}_X F = 0$. A lightlike submanifold $M$ is called **Lie recurrent** if it admits a Lie recurrent structure tensor field $F$.

**Theorem 4.3.** Let $M$ be a Lie recurrent generic lightlike submanifold of an indefinite trans-Sasakian manifold $\tilde{M}$ with a quarter-symmetric metric connection. Then the following statements are satisfied:

1. $F$ is Lie parallel,
2. $\alpha = 0$ and $d\vartheta = 0$. Thus $\tilde{M}$ is not an indefinite Sasakian manifold,
3. $h^i_\ell$ is never symmetric on $S(TM)$,
4. $\tau_{ij}$ and $\rho_{ia}$ are satisfied $\tau_{ij} \circ F = 0$ and $\rho_{ia} \circ F = 0$. Moreover,

$$ \tau_{ij}(X) = \sum_{k=1}^r u_i(X) g(A_{N_k} V_j, N_i) - \beta \delta_{ij} \theta(X). $$

**Proof.** (1) As $(\mathcal{L}_X F)Y = [X, FY] - F[X, Y]$, using (3.6) and (3.20), we get

$$ \vartheta(X) V_j = \nabla_{V_j} X + F\nabla_{\xi_j} X + \beta u_j(X) \zeta $$

$$ + \sum_{i=1}^r u_i(Y) A_{N_i} X + \sum_{a=r+1}^n w_a(Y) A_{E_a} X $$

$$ - \sum_{i=1}^r \{ h^i_\ell(X, Y) - \theta(Y) u_i(X) \} U_i $$

$$ - \sum_{a=r+1}^n \{ h^a_s(X, Y) - \theta(Y) w_a(X) \} W_a $$

$$ + \alpha \{ g(X, Y) \zeta - \theta(Y) X \} + \beta \{ g(JX, Y) \zeta - \theta(Y) FX \}, $$

by (3.5). Replacing $Y$ by $\xi_j$ and then, $Y$ by $V_j$ to (4.15) by turns, we have

$$ -\vartheta(X) V_j = \nabla_{V_j} X + F\nabla_{\xi_j} X + \beta u_j(X) \zeta $$

$$ - \sum_{i=1}^r h^i_\ell(X, \xi_j) U_i - \sum_{a=r+1}^n h^a_s(X, \xi_j) W_a, $$

(4.16)

$$ -\vartheta(X) \xi_j = -\nabla_{\xi_j} X + F\nabla_{V_j} X + \alpha u_j(X) \zeta $$

(4.17)
respectively. Taking the scalar product with $U_j$ to (4.16) and then, taking the scalar product with $N_j$ to (4.17), we obtain respectively

$$-\vartheta(X) = g(\nabla_{V_j} X, U_j) - \bar{g}(\nabla_{\xi_j} X, N_j),$$
$$\vartheta(X) = g(\nabla_{V_j} X, U_j) - \bar{g}(\nabla_{\xi_j} X, N_j).$$

Comparing these two equations, we get $\vartheta = 0$. Thus $F$ is Lie parallel.

(2) Taking the scalar product with $\zeta$ to (4.17) satisfying $\vartheta = 0$, we have

$$g(\nabla_{\xi_j} X, \zeta) = \alpha u_j(X).$$

Replacing $X$ by $U_j$ to this equation and using (3.17), we obtain $\alpha = 0$.

Applying $\nabla_X$ to $\theta(\bar{Y}) = g(\bar{Y}, \zeta)$ and using (1.1) and (2.3), we obtain

$$d\theta(\bar{X}, \bar{Y}) = \alpha g(\bar{X}, J\bar{Y}),$$

due to the fact that $\bar{\nabla}$ is metric. As $\alpha = 0$, we see that $d\theta = 0$.

(3) Replacing $X$ by $U_i$ to (4.15) and using (3.2), (3.3), (3.5), (3.7), (3.8), (3.11), (3.15), (3.16), and (3.17), we obtain

$$(4.18) \quad \sum_{k=1}^{r} u_k(Y) A_{n_k} U_i + \sum_{a=r+1}^{n} w_a(Y) A_{n_a} U_i - \theta(Y) U_i + \beta \eta_a(Y) \zeta$$
$$- A_{n_i} Y - F(A_{n_i} F Y) - \sum_{j=1}^{r} \tau_{ij}(F Y) U_j - \sum_{a=r+1}^{n} \rho_{ia}(F Y) W_a = 0.$$

Taking $Y = \zeta$ to (4.18) and then, taking the scalar product with $PX$, we get $h^*_i(\zeta, PX) = -v_i(PX)$. Assume that $h^*_i$ is symmetric on $S(TM)$. Taking $X = PX$ to (3.15), we obtain $h^*_i(\zeta, PX) = 0$. It follows that $v_i(PX) = 0$. It is a contradiction to $v_i(V_i) = 1$. Thus $h^*_i$ is never symmetric on $S(TM)$.

(4) Taking the scalar product with $N_i$ to (4.16) such that $X = W_a$ and using (3.8), (3.10), (3.12), and (3.19), we get $h^*_i(U_i, V_j) = \rho_{ia}(\xi_j)$. On the other hand, taking the scalar product with $W_a$ to (4.17) such that $X = U_i$ and using (3.17), we have $h^*_a(U_i, V_j) = -\rho_{ia}(\xi_j)$. Thus $\rho_{ia}(\xi_j) = 0$ and $h^*_a(U_i, V_j) = 0$.

Taking the scalar product with $U_i$ to (4.16) such that $X = W_a$ and using (3.10), (3.12), and (3.19), we get $\epsilon_a \rho_{ia}(V_j) = \phi_{ai}(U_i)$. On the other hand, taking the scalar product with $W_a$ to (4.16) such that $X = U_i$ and using (3.12) and (3.17), we get $\epsilon_a \rho_{ia}(V_j) = -\phi_{aj}(U_i)$. Thus $\rho_{ia}(V_j) = 0$ and $\phi_{aj}(U_i) = 0$.

Taking the scalar product with $V_i$ to (4.16) such that $X = W_a$ and using (3.7), (3.8), (3.12), and (3.14), we get $\phi_{ai}(V_j) = -\phi_{aj}(V_i)$. On the other hand, taking the scalar product with $W_a$ to (4.16) such that $X = V_i$ and using (3.12) and (3.18), we have $\phi_{ai}(V_j) = \phi_{aj}(V_i)$. Thus $\phi_{ai}(V_j) = 0$.

Taking the scalar product with $W_a$ to (4.16) such that $X = \xi_i$ and using (2.3), (3.9) and (3.12), we get $h^*_i(V_j, W_a) = \phi_{ai}(\xi_j)$. On the other hand, taking the scalar product with $V_i$ to (4.17) such that $X = W_a$ and using (3.7) and
(3.19), we have \( h'_t(V'_j, W_a) = -\phi_{ai}(\xi_j) \). Thus \( \phi_{ai}(\xi_j) = 0 \) and \( h'_t(V'_j, W_a) = 0 \).

Summarizing the above results, we obtain

\[
\begin{align*}
\rho_a(\xi_j) = 0, & \quad \rho_a(V'_j) = 0, & \quad \phi_{ai}(V'_j) = 0, & \quad \phi_{ai}(V'_j) = 0, & \quad h'_a(U_i, V'_j) = h'_f(U_i, W_a) = 0, & \quad h'_f(V'_j, W_a) = h'_a(V'_j, V'_j) = 0.
\end{align*}
\]

Taking the scalar product with \( N_i \) to (4.15) and using (3.12), we have

\[
\begin{align*}
(4.20) & \quad -\bar{g}(\nabla F X, N_i) + \bar{g}(\nabla X, U_i) - \theta(Y)\{\eta_i(X) + \beta v_i(X)\} \\
& + \sum_{k=1}^{r} u_k(Y)\bar{g}(A_{N_k} X, N_i) + \sum_{a=r+1}^{n} \epsilon_{a}w_{a}(Y)\rho_{a}(X) = 0.
\end{align*}
\]

Replacing \( X \) by \( V'_j \) to (4.20) and using (3.9), (3.18) and (4.19), we have

\[
(4.21) \quad h'_f(F X, U_i) + \tau_{ij}(X) + \beta \delta_{ij} \theta(X) = \sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k} V'_j, N_i).
\]

Replacing \( X \) by \( \xi_j \) to (4.20) and using (2.8), (3.9) and (4.19), we have

\[
(4.22) \quad h'_f(X, U_i) + \delta_{ij} \theta(X) = \sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k} \xi_j, N_i).
\]

Taking \( X = U_k \) to (4.22), we have

\[
(4.23) \quad h'_a(U_k, V'_j) = h'_f(U_k, U_i) = \bar{g}(A_{N_k} \xi_j, N_i).
\]

On the other hand, taking the scalar product with \( V'_j \) to (4.18) and using (3.11), (3.12), (3.16), (4.19) and (4.23), we get

\[
\begin{align*}
& \quad h'_f(X, U_i) + \delta_{ij} \theta(X) = -\sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k} \xi_j, N_i) - \tau_{ij}(FX).
\end{align*}
\]

Comparing this equation with (4.22), we obtain

\[
\tau_{ij}(FX) + \sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k} \xi_j, N_i) = 0.
\]

Replacing \( X \) by \( U_h \) to this equation, we have \( \bar{g}(A_{N_k} \xi_j, N_i) = 0 \). Therefore,

\[
(4.24) \quad \tau_{ij}(FX) = 0, \quad h'_f(X, U_i) + \delta_{ij} \theta(X) = 0.
\]

Taking \( X = FY \) to (4.24) and (4.25), we get \( h'_f(FX, U_i) = 0 \). Thus (4.21) is reduced to

\[
(4.25) \quad \tau_{ij}(X) = \sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k} V'_j, N_i) - \beta \delta_{ij} \theta(X).
\]

Replacing \( Y \) by \( W_b \) to (4.18), we have \( A_{N_b} U_i = A_{N_i} W_a \). Taking the scalar product with \( U_j \) and using (3.8), (3.10), (3.11) and (3.16), we have

\[
(4.26) \quad h'_a(W_a, U_j) = \epsilon_{a}h'_a(U_i, U_j) = \epsilon_{a}h'_a(U_j, U_i) = h'_a(U_j, W_a).
\]
Taking the scalar product with $W_a$ to (4.18), we have
\[
\epsilon_a \rho_{\alpha a}(FY) = -h^a_i(Y, W_a)
\]
\[
+ \sum_{k=1}^r u_k(Y)h^a_k(U_i, W_a) + \sum_{b=r+1}^n \epsilon_b w_b(Y)h^a_b(U_i, W_a).
\]

Taking the scalar product with $U_i$ to (4.15) and then, taking $X = W_a$ and using (3.8), (3.10), (3.11), (3.12), (3.16), (3.19) and (4.26), we obtain
\[
\epsilon_a \rho_{\alpha a}(FY) = h^a_i(Y, W_a)
\]
\[
- \sum_{k=1}^r u_k(Y)h^a_k(U_i, W_a) - \sum_{b=r+1}^n \epsilon_b w_b(Y)h^a_b(U_i, W_a).
\]
Comparing the last two equations, we obtain $\rho_{\alpha a}(FY) = 0$. \hfill \square

**Remark 4.4.** Replacing $X$ by $\xi_j$ to (3.9) and using (3.12), we have
\[
h^a_i(\xi_j, X) = g(A^*_a \xi_j, X).
\]
Taking $Y = \xi_j$ to (3.7), we obtain $h^a_i(X, \xi_j) = h^a_j(\xi_j, X)$. From this and (3.12), we see that $h^a_i(\xi_j, X)$ are skew-symmetric with respect to $i$ and $j$. It follows that $A^*_a \xi_j = -A^*_a \xi_i$, i.e., $A^*_a \xi_j$ are skew-symmetric with respect to $i$ and $j$.

In case $M$ is Lie recurrent, taking $Y = U_j$ to (4.18), we have $A_{\alpha i} U_j = A_{\alpha j} U_i$. Thus $A_{\alpha i} U_j$ are symmetric with respect to $i$ and $j$. Therefore, we get
\[
h^a_i(\xi_j, F(A_{\alpha j} U_i)) = g(A^*_a \xi_j, F(A_{\alpha j} U_i)) = 0,
\]
\[
h^a_i(\xi_j, W_a) = \epsilon_a h^a_i(\xi_j, V_i) = \epsilon_a h^a_i(V_i, \xi_j) = -\phi_{ji}(V_i) = 0,
\]
due to (4.19). Taking $X = U_i$ (3.7) and using (4.24), we obtain
\[
h^a_j(U_i, X) = 0.
\]

### 5. Indefinite generalized Sasakian space forms

Alegre and his collaborators [1] introduced generalized Sasakian space form. Jin [6] extended this notion as follow: An indefinite trans-Sasakian manifold $\bar{M}$ is called indefinite generalized Sasakian space form and denoted by $\bar{M}(f_1, f_2, f_3)$ if there exist three smooth functions $f_1, f_2$ and $f_3$ on $\bar{M}$ such that
\[
\bar{R}(X, \bar{Y})\bar{Z} = f_1(\bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y}) + f_2(\bar{g}(\bar{X}, J\bar{Z})J\bar{Y} - \bar{g}(\bar{Y}, J\bar{Z})J\bar{X}) + 2\bar{g}(\bar{X}, J\bar{Y})J\bar{Z}
\]
\[
+ f_3(\theta(\bar{X})\theta(\bar{Z})\bar{Y} - \theta(\bar{Y})\theta(\bar{Z})\bar{X}) + \bar{g}(\bar{X}, \bar{Z})\theta(\bar{Y})\theta(\bar{X})\bar{Z} - \bar{g}(\bar{Y}, \bar{Z})\theta(\bar{X})\bar{Z},
\]
where the symbol $\bar{R}$ is the curvature tensor of $\bar{M}(f_1, f_2, f_3)$.

Sasakian space form, Kenmotsu space form and cosymplectic space form are important kinds of generalized Sasakian space forms such that
\[
f_1 = \frac{c+1}{c}, f_2 = f_3 = \frac{c-1}{c}; \quad f_1 = \frac{c-1}{c}, f_2 = f_3 = \frac{c+1}{c}; \quad f_1 = f_2 = f_3 = \frac{c}{c}.
\]
respectively, where $c$ is a constant J-sectional curvature of each space form.

Denote by $\bar{R}$, $R$ and $R^*$ the curvature tensors of the quart-symmetric metric connection $\bar{\nabla}$ on $\bar{M}$, and the induced connection $\nabla$ and $\nabla^*$ on $M$ and $S(TM)$ respectively. Using the Gauss-Weingarten formulas for $M$ and $S(TM)$, we obtain the Gauss equations for $M$ and $S(TM)$, respectively:

$\bar{R}(X, Y)Z = R(X, Y)Z$

$$+ \sum_{i=1}^{r} \{h^J_i(X, Z)A_{N_i}Y - h^J_i(Y, Z)A_{N_i}X\}$$

$$+ \sum_{a=r+1}^{n} \{h_{a}^{*}(X, Z)A_{E_{a}}Y - h_{a}^{*}(Y, Z)A_{E_{a}}X\}$$

$$+ \sum_{i=1}^{r} \{((\nabla_{X}h^{J}_i)(Y, Z) - (\nabla_{Y}h^{J}_i)(X, Z)$$

$$+ \sum_{j=1}^{r} \{r_{ji}(X)h^{J}_j(Y, Z) - r_{ji}(Y)h^{J}_j(X, Z)\}$$

$$+ \sum_{a=r+1}^{n} \{[\phi_{a}(X)h_{a}^{*}(Y, Z) - \phi_{a}(Y)h_{a}^{*}(X, Z)] - \theta(X)h_{a}^{*}(FY, Z) + \theta(Y)h_{a}^{*}(FX, Z)\}N_{i}$$

$$+ \sum_{a=r+1}^{n} \{(\nabla_{X}h_{a}^{*})(Y, Z) - (\nabla_{Y}h_{a}^{*})(X, Z)$$

$$+ \sum_{i=1}^{r} \{[\rho_{a}(X)h^{J}_i(Y, Z) - \rho_{a}(Y)h^{J}_i(X, Z)]$$

$$+ \sum_{b=r+1}^{n} \{[\sigma_{b}(X)h_{b}^{*}(Y, Z) - \sigma_{b}(Y)h_{b}^{*}(X, Z)]$$

$$- \theta(X)h_{b}^{*}(FY, Z) + \theta(Y)h_{b}^{*}(FX, Z)\}E_{a},$$

(5.2)

$R(X, Y)PZ = R^{*}(X, Y)PZ$

$$+ \sum_{i=1}^{r} \{h^J_i(X, PZ)A_{E_{i}}Y - h^J_i(Y, PZ)A_{E_{i}}X\}$$

$$+ \sum_{i=1}^{r} \{((\nabla_{X}h^{J}_i)(Y, PZ) - (\nabla_{Y}h^{J}_i)(X, PZ)$$

$$+ \sum_{j=1}^{r} \{[h^{J}_j(X, PZ)\tau_{ij}(Y) - h^{J}_j(Y, PZ)\tau_{ij}(X)]$$

$$- \theta(X)h^{J}_j(FY, Z) + \theta(Y)h^{J}_j(FX, Z)\}E_{j},$$

(5.3)
Comparing the tangential and lightlike transversal components of the two equations (5.1) and (5.2), and using (3.4), we get

\[ R(X, Y)Z = f_1 \{ g(Y, Z)X - g(X, Z)Y \} + f_2 \{ \bar{g}(X, JZ)FY - \bar{g}(Y, JZ)FX + 2\bar{g}(X, JY)FZ \} + f_3 \{ \theta(X)\theta(Z)Y - \theta(Y)\theta(Z)X \} \]

\[ + \bar{g}(X, Z)\theta(Y)\zeta - \bar{g}(Y, Z)\theta(X)\zeta \]

\[ + \sum_{i=1}^{r} \{ h_i^i(Y, Z)A_{\eta_i}X - h_i^i(X, Z)A_{\eta_i}Y \} \]

\[ + \sum_{a=r+1}^{n} \{ h_a^i(Y, Z)A_{\eta_a}X - h_a^i(X, Z)A_{\eta_a}Y \} \]

Taking the scalar product with \( N_i \) to (5.3), we have

\[ \bar{g}(R(X, Y)PZ, N_i) = (\nabla_X h_i^i)(Y, Z) - (\nabla_Y h_i^i)(X, Z) \]

\[ + \sum_{j=1}^{r} \{ \tau_{ji}(X)h_j^i(Y, Z) - \tau_{ji}(Y)h_j^i(X, Z) \} \]

\[ + \sum_{a=r+1}^{n} \{ \phi_{ai}(X)h_a^i(Y, Z) - \phi_{ai}(Y)h_a^i(X, Z) \} \]

\[ - \theta(X)h_i^i(FY, Z) + \theta(Y)h_i^i(FX, Z) \]

\[ = f_2 \{ u_i(Y)\bar{g}(X, JZ) - u_i(X)\bar{g}(Y, JZ) + 2u_i(Z)\bar{g}(X, JY) \}. \]

Substituting (5.4) into the last equation and using (3.12)4, we obtain

\[ (\nabla_X h_i^i)(Y, PZ) - (\nabla_Y h_i^i)(X, PZ) \]

\[ + \sum_{j=1}^{r} \{ \tau_{ij}(Y)h_j^i(X, PZ) - \tau_{ij}(X)h_j^i(Y, PZ) \} \]

\[ - \theta(X)h_i^i(FY, Z) + \theta(Y)h_i^i(FX, Z) \]

\[ = f_1 \{ g(Y, PZ)\eta_i(X) - g(X, PZ)\eta_i(Y) \}. \]
Substituting this into (5.5) such that replace
Let
Theorem 5.1. Let \( M \) be a generic lightlike submanifold of an indefinite generalized Sasakian space form \( M(\alpha, f_1, f_2, f_3) \) with a quarter-symmetric metric connection. Then the following properties are satisfied

1. \( \alpha \) is a constant,
2. \( \alpha \beta = 0 \),
3. \( f_1 - f_2 = \alpha^2 - \beta^2 \) and \( f_1 - f_3 = (\alpha^2 - \beta^2) + \alpha - \zeta \beta \).

Proof. Applying \( \nabla_X \) to \((3.16)_1\): \( h^f_{ij}(Y, U_j) = h^*_{ij}(Y, V_j) \) and using \((2.1), (3.2), (3.3), (3.4), (3.9), (3.11), (3.16)_1, (3.17) \) and \((3.18) \), we have

\[
\begin{align*}
(\nabla_X h^f_{ij})(Y, U_i) &= (\nabla_X h^*_i)(Y, V_j) \\
&= -\sum_{k=1}^r \{\tau_{kj}(X)h^f_{ik}(Y, U_i) + \tau_{ik}(X)h^f_{kj}(Y, V_j)\} \\
&\quad - \sum_{a=r+1}^n \{\phi_{aj}(X)h^*_a(Y, U_i) + \epsilon_{a\theta}(X)h^*_a(Y, V_j)\} \\
&\quad + \sum_{k=1}^r \{h^*_i(Y, U_k)h^f_{kj}(X, \xi_j) + h^*_i(X, U_k)h^f_{kj}(Y, \xi_j)\} \\
&\quad - g(A^*_k, X, F(A, Y)) + g(A^*_k, Y, F(A, X)) \\
&\quad - \sum_{k=1}^r h^f_{ij}(X, V_k)\eta_k(A, Y) - \alpha^2 u_j(Y)\eta_i(X) \\
&\quad - \beta^2 u_j(X)\eta_i(Y) + \alpha\beta\{u_j(X)v_i(Y) - u_j(Y)v_i(X)\}.
\end{align*}
\]

Substituting this into (5.5) such that replace \( i \) by \( j \) and take \( Z = U_i \), we have

\[
\begin{align*}
(\nabla_X h^*_j)(Y, V_j) &= (\nabla_Y h^*_j)(X, V_j) \\
&= \sum_{k=1}^r \{\tau_{ik}(X)h^f_{kj}(Y, V_j) - \tau_{jk}(Y)h^f_{ik}(X, V_j)\} \\
&\quad - \sum_{a=r+1}^n \epsilon_a\{h^*_a(Y, V_j)\rho_a(X) - h^*_a(X, V_j)\rho_a(Y)\} \\
&\quad - \sum_{k=1}^r \{h^f_{kj}(Y, V_j)\eta_k(A, X) - h^f_{kj}(X, V_j)\eta_k(A, Y)\} \\
&\quad - \theta(X)h^*_j(FY, V_j) + \theta(Y)h^*_j(FX, V_j) \\
&\quad + (\alpha^2 - \beta^2)\{u_j(X)\eta_i(Y) - u_j(Y)\eta_i(X)\} \\
&\quad + 2\alpha\beta\{u_j(X)v_i(Y) - u_j(Y)v_i(X)\} \\
&= f_2\{u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y) + 2\delta_{ij}\theta(X, JY)\}.
\end{align*}
\]
Thus $\alpha Y$

Replacing Substituting this and (3.16) into (5.5) such that $Z$

Taking $X = \xi_i$ and $Y = U_j$, and $X = V_i$ and $Y = U_j$ by turns, we have

$f_1 - f_2 = \alpha^2 - \beta^2, \quad \alpha \beta = 0$.

Applying $\nabla_X$ to $\eta_i(Y) = \bar{g}(Y, N_i)$ and using (2.5), we have

$(\nabla_X \eta_i)Y = -g(A_{\alpha}, X, Y) + \sum_{j=1}^{r} \tau_{ij}(X)\eta_j(Y)$.

Applying $\nabla_Y$ to (3.16) and using (3.13) and (3.22), we have

$(\nabla_X h_i^s)(Y, \zeta) = -(X\alpha)v_i(Y) + (X\beta)\eta_i(Y) + \alpha^2\theta(Y)\eta_i(X) + \beta^2\theta(Y)\eta_i(Y) + \alpha\{g(A_{\alpha}, X, FY) + g(A_{\alpha}, Y, F X) - \sum_{j=1}^{r} v_j(Y)\tau_{ij}(X)\}

- \sum_{a=r+1}^{n} \epsilon_a u_a(Y)\rho_u(X) - \sum_{j=1}^{r} u_j(Y)\eta_i(A_{\alpha} X)\}

- \beta\{g(A_{\alpha}, X, Y) + g(A_{\alpha}, Y, X) - \sum_{j=1}^{r} \tau_{ij}(X)\eta_j(Y)\}$.

Substituting this and (3.16) into (5.6) such that $PZ = \zeta$, we get

\[
{X\alpha + [f_1 - f_3 - (\alpha^2 - \beta^2) - \alpha]\theta(X)}\eta_i(Y)

- \{Y\beta + [f_1 - f_3 - (\alpha^2 - \beta^2) - \alpha]\theta(Y)}\eta_i(X)

= (X\alpha)v_i(Y) - (Y\alpha)v_i(X).
\]

Taking $X = \zeta$ and $Y = \xi_i$, and taking $X = U_h$ and $Y = V_i$ by turns, we get

$f_1 - f_3 = (\alpha^2 - \beta^2) + \alpha - \zeta \beta, \quad U_i \alpha = 0, \quad \forall i$.

Applying $\nabla_X$ to $h_i^s(Y, \zeta) = -\alpha u_i(Y)$ and using (3.21) and (3.13), we get

$(\nabla_X h_i^s)(Y, \zeta) = -(X\alpha)u_i(Y) + \alpha\{\sum_{j=1}^{r} u_j(Y)\tau_{ij}(X) + \sum_{a=r+1}^{n} w_a(Y)\phi_a(X)\}

+ h_i^s(X, FY) + h_i^s(Y, FX)$.

Substituting this and (3.16) into (5.5) such that $Z = \zeta$, we obtain

$(X\alpha)u_i(Y) = (Y\alpha)u_i(X)$.

Replacing $Y$ by $U_i$ to this equation, we obtain $X\alpha = 0$ for all $X \in \Gamma(TM)$. Thus $\alpha$ is a constant. This completes the proof of the theorem.

We say that $\bar{M}$ (resp. $M$) is flat if $\bar{R} = 0$ (resp. $R = 0$).
**Theorem 5.2.** Let \( M \) be a recurrent generic lightlike submanifold of an indefinite generalized Sasakian space form \( \bar{M}(f_1, f_2, f_3) \) with a quarter-symmetric metric connection. Then \( \bar{M}(f_1, f_2, f_3) \) is flat.

**Proof.** As \( M \) is recurrent, by Theorem 4.2, we get (4.10), (4.11), (4.12) and the results: \( \alpha = \beta = 0 \) and \( \rho_a = 0 \). As \( \alpha = \beta = 0 \), \( f_1 = f_2 = f_3 \) by Theorem 5.1. Taking the scalar product with \( N_j \), \( U_j \) and \( W_a \) to (4.10), by turns, we get

\[
\eta_j(A_{X_j} X) = 0, \quad h_\theta(X, U_j) = 0, \quad h^{\theta}_{\alpha}(X, U_i) = h^{\theta}_{\alpha}(X, W_a) = 0.
\]

Applying \( \nabla_X \) to \( h^{\theta}_{\alpha}(Y, U_j) = 0 \) and using (4.12), we obtain

\[
(\nabla_X h^{\theta}_{\alpha})(Y, U_j) = 0.
\]

Taking \( PZ = U_j \) to (5.6) and using the last two equations, we have

\[
f_1\{v_j(Y)\eta_i(X) - v_i(X)\eta_j(Y)\} + f_2\{v_i(Y)\eta_j(X) - v_j(X)\eta_i(Y)\} = 0.
\]

Taking \( X = \xi_i \) and \( Y = V_j \) to this equation, we have \( f_1 = 0 \). It follows that \( f_1 = f_2 = f_3 = 0 \) and \( \bar{M}(f_1, f_2, f_3) \) is flat. \( \Box \)

**Theorem 5.3.** Let \( M \) be a generic lightlike submanifold of \( \bar{M}(f_1, f_2, f_3) \) with a quarter-symmetric metric connection. If \( M \) is Lie recurrent, then \( \bar{M}(f_1, f_2, f_3) \) is a space form with an indefinite \( \beta \)-Kenmotsu structure such that

\[
f_1 = -\beta^2, \quad f_2 = 0, \quad f_3 = \zeta \beta.
\]

**Proof.** Applying \( \nabla_X \) to (4.24)2: \( h_\theta(X, U_j) = -\delta_j \theta(Y) \), we have

\[
(\nabla_X h_\theta)(Y, U_j) = -\delta_j \{X(\theta(Y)) - \theta(\nabla_X Y)\} - h_\theta(Y, \nabla_X U_j).
\]

Using this equation, (3.6), (3.14), (4.24)2 and the facts that \( \alpha = 0 \), \( d\theta = 0 \) and \( \theta(FX) = 0 \), we have

\[
(\nabla_X h_\theta)(Y, U_j) - (\nabla_Y h_\theta)(X, U_j)
\]

\[
= h_\theta(X, F(A_{X_j} Y)) - \tau_{ji}(Y)\theta(X) + \sum_{a=\ell+1}^n \rho_{ja}(Y)h_\theta(X, W_a)
\]

\[
- h_\theta(Y, F(A_{X_j} X)) + \tau_{ij}(X)\theta(Y) - \sum_{a=\ell+1}^n \rho_{ja}(X)h_\theta(Y, W_a).
\]

Replacing \( Z \) by \( U_j \) to (5.5) and using (4.24)2 and \( \theta(FX) = 0 \), we obtain

\[
h_\theta(X, F(A_{X_j} Y)) - h_\theta(Y, F(A_{X_j} X))
\]

\[
+ \sum_{a=\ell+1}^n \{\rho_{ja}(Y)h_\theta(X, W_a) - \rho_{ja}(X)h_\theta(Y, W_a)\}
\]

\[
+ \sum_{a=\ell+1}^n \{\phi_{aj}(X)h_\theta(X, U_j) - \phi_{aj}(Y)h_\theta(X, U_j)\}
\]

\[
= f_2\{u_i(Y)\eta_j(X) - u_i(X)\eta_j(Y)\} + 2\delta_{ij}\bar{g}(X, JY).
\]
Replacing $Y = U_i$ and $X = \xi_j$ to this equation and using (4.19), (4.27), (4.28) and (4.29), we have $f_2 = 0$. As $f_2 = 0$, we have $f_1 = -\beta^2$ and $f_3 = \zeta \beta$. \hfill \Box

**Theorem 5.4.** Let $M$ be a generic lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection. If $U_i$s are parallel with respect to $\nabla$, then $\tau_{ij} = 0$, $\bar{M}$ is an indefinite cosymplectic manifold and $M$ is solenoidal. Moreover, if $\bar{M} = \bar{M}(f_1, f_2, f_3)$, then it is flat.

**Proof.** If $U_i$ is parallel with respect to $\nabla$, then, taking the scalar product with $\zeta, V_j, W_a, U_j$ and $N_j$ to (3.17) such that $\nabla_X U_i = 0$ by turns, we get

\[(5.7) \quad \alpha = \beta = 0, \quad \tau_{ij} = 0, \quad \rho_{i\alpha} = 0, \quad \eta_j(A_{N_i} X) = 0, \quad h^*_i(X, U_j) = 0,\]

respectively. As $\alpha = \beta = 0$, $\bar{M}$ is an indefinite cosymplectic manifold. As $\rho_{i\alpha} = 0$ and $\eta_j(A_{N_i} X) = 0$, $M$ is solenoidal.

As $\alpha = \beta = 0$, $f_1 = f_2 = f_3$ by Theorem 5.1. Applying $\nabla_Y$ to (5.7)$_5$ and using (5.7)$_5$ and the fact that $\nabla_X U_i = 0$, we obtain

\[(\nabla_X h^*_i)(Y, U_j) = 0.\]

Substituting this equation and (5.7) into (5.6) with $PZ = U_j$, we have

\[f_1\{v_i(Y)\eta_h(X) - v_j(X)\eta_h(Y)\} + f_2\{v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)\} = 0.\]

Taking $X = \xi_i$ and $Y = V_j$ to this equation, we obtain $f_1 = 0$. Therefore, $f_1 = f_2 = f_3 = 0$ and $\bar{M}(f_1, f_2, f_3)$ is flat. \hfill \Box

**Theorem 5.5.** Let $M$ be a generic lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection. If $V_i$s are parallel with respect to $\nabla$, then $\tau_{ij} = 0$, $\alpha = -1$ and $\beta = 0$, i.e., $\bar{M}$ is an indefinite Sasakian manifold, and $\phi_{ai} = h^*_i(X, \xi_j) = 0$, i.e., $M$ is irrotational. Moreover, if $\bar{M} = \bar{M}(f_1, f_2, f_3)$, then $\bar{M}(f_1, f_2, f_3)$ is a space form with an indefinite Sasakian structure of the curvature functions

\[f_1 = f_3 = \frac{2}{3}, \quad f_2 = -\frac{1}{3}.\]

**Proof.** If $V_i$ is parallel with respect to $\nabla$, then, taking the scalar product with $\zeta, U_j, V_j, W_a$ and $N_j$ to (3.18) with $\nabla_V V_i = 0$ by turns, we get respectively

\[(5.8) \quad \beta = 0, \quad \tau_{ij} = 0, \quad h^*_i(X, \xi_i) = 0, \quad \phi_{ai} = 0, \quad h^*_i(X, U_j) = 0\]

and we have $F(A^*_a X) = 0$. As $h^*_i(X, \xi_i) = 0$ and $\phi_{ai} = 0$, $M$ is irrotational. Replacing $Y$ by $\xi_j$ and $U_j$ to (3.7) by turns and using (5.8)$_3, 5$, we have

\[(5.9) \quad h^*_i(\xi_j, X) = 0, \quad h^*_i(U_j, X) = \delta_{ij}\theta(X).\]

Taking $X = U_i$ to (3.14)$_1$ and using (5.9)$_2$, we get

\[-\alpha = -\alpha u_i(U_i) = h^*_i(U_i, \zeta) = \theta(\zeta) = 1.\]

As $\alpha = -1$ and $\beta = 0$, $\bar{M}$ is an indefinite Sasakian manifold.

Applying $\nabla_X$ to (5.8)$_5$ and using (3.4), (3.14)$_1$, (3.17) and (5.8)$_3$, we have

\[(\nabla_X h^*_i)(Y, U_j) = h^*_i(Y, V_k)g(A_{N_j} X, N_k).\]
Taking $\gamma$ if there exist smooth functions $\rho_ja(X)h_i^j(Y, W_a) - u_i(Y)\eta_j(X)$. Substituting the last two equations into (5.5) with $Z = U_j$, we obtain

$$h^i_j(Y, V_k)g(A_{\alpha_j}X, N_k) - h^i_j(X, V_k)g(A_{\alpha_j}Y, N_k) + u_i(X)\eta_j(Y) - u_i(Y)\eta_j(X) + \sum_{a=r+1}^n \rho_ja(Y)h_i^j(X, W_a) - \rho_ja(X)h_i^j(Y, W_a)$$

$$= f_2\{u_i(Y)\eta_j(X) - u_i(X)\eta_j(Y) + 2\delta_{ij}\bar{g}(X, JY)\}.$$ Taking $X = \xi_i$ and $Y = U_i$ to this and using (5.9), we obtain $3f_2 = -1$. As $f_2 = -\frac{1}{3}$, we have $f_1 = f_3 = \frac{2}{3}$ by Theorem 5.1. \qed

**Definition.** A screen distribution $S(TM)$ is called *totally umbilical* [4] in $M$ if there exist smooth functions $\gamma_i$ such that $A_{\alpha_j}X = \gamma_iPX$, or equivalently,

$$h^*_i(X, PY) = \gamma_i\bar{g}(X, Y).$$

In case $\gamma_i = 0$ for all $i$, we say that $S(TM)$ is *totally geodesic* in $M$.

**Theorem 5.6.** Let $M$ be a generic lightlike submanifold of $M(f_1, f_2, f_3)$ with a quarter-symmetric metric connection. If $S(TM)$ is totally umbilical in $M$, then $\bar{M}(f_1, f_2, f_3)$ is flat and $S(TM)$ is totally geodesic.

**Proof.** Assume that $S(TM)$ is totally umbilical. Then (3.17) is reduced to $\gamma_i\theta(X) = -\alpha\gamma_i(X) + \beta\eta_i(X)$ for all $i$. Replacing $X$ by $V_i$, $\xi_i$ and $\zeta$ to this equation by turns, we have $\alpha = \beta = \gamma_i = 0$. As $\gamma_k = 0$, $S(TM)$ is totally geodesic. As $\alpha = 0$, $f_1 = f_2 = f_3$ by Theorem 5.1. Taking $PZ = U_k$ to (5.6) with $h^*_i = 0$ and using the facts that $h^*_i(X, U_k) = h^*_i(X, W_a) = 0$ and $h^*_i(X, V_j) = h^*_i(X, V_j) = 0$, we get

$$f_1\{u_k(Y)\eta_i(X) - u_k(X)\eta_i(Y)\} + f_2\{v_i(Y)\eta_k(X) - v_i(X)\eta_k(Y)\} = 0.$$ Taking $X = \xi_i$ and $Y = V_k$ to this equation, we get $f_1 = 0$. Thus $f_1 = f_2 = f_3 = 0$ and $\bar{M}(f_1, f_2, f_3)$ is flat. \qed

**References**


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